UMTYMP Geometry Day 11

Chapter 12 Circles and Angles

Chapter 13 Power of a Point

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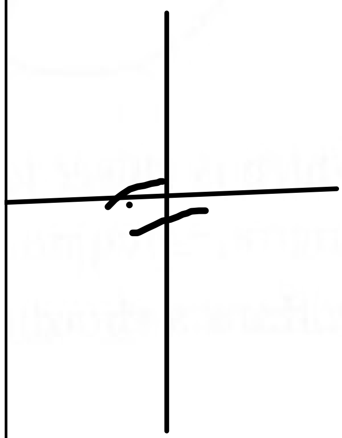
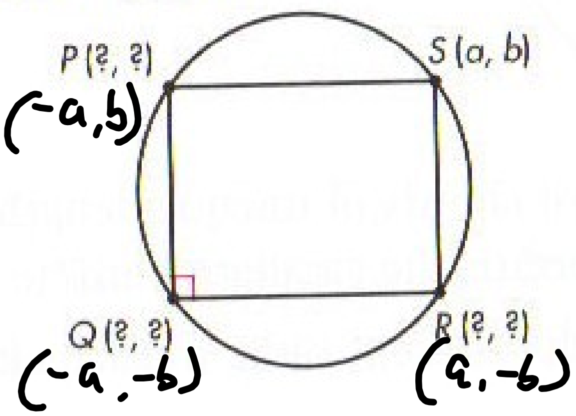
ribed
. Find
and O.

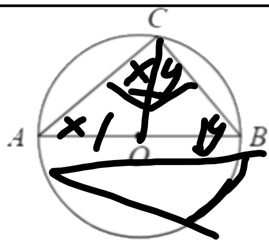
17. Polygon $PQRS$ is a rectangle inscribed in a circle centered at the origin. The slope of $\overline{PS}$ is 0. Find the coordinates of points P, Q, and R.





Circles and Angles

These are some of my favorite problems! :-)



$x + x + y + y = 180$

$2x + 2y = 180$

$x + y = 90$

In Section 7.2 **here**, we learned that the circumcenter of a right triangle is the midpoint of its hypotenuse. Thus, when we draw the circumcircle of a right triangle, the right angle is inscribed in a semicircle. In this problem, we investigate if it is true that any angle inscribed in a semicircle, such as $\angle ACB$ in the diagram at right, must be a right angle. Let O be the center of the circle.

(a) Draw OC. What do we know about $\triangle AOC$ and $\triangle BOC$?

(b) Let $\angle A = x$ and $\angle B = y$. What other angle measures x degrees? What other angle measures y degrees?

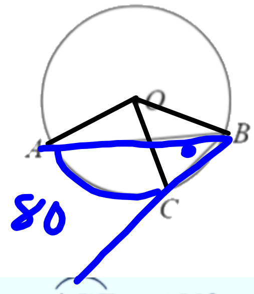
(c) Consider the sum of the angles in $\triangle ABC$ to show that $x + y = 90°$.

(d) Must $\angle ACB$ be a right angle?

angles inscribed in a semi-circle $= 90°$

Central Angles:

Inscribed angles are ½ the arc they intercept.



In this problem we will find inscribed $\angle ABC$ shown at right, given that $\overset{\frown}{AC} = 80^\circ$ and $\overset{\frown}{ACB} = 130^\circ$.

(a) Draw $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$. Find $\angle AOC$ and $\angle COB$.

(b) Find $\angle OBC$ and $\angle OBA$.

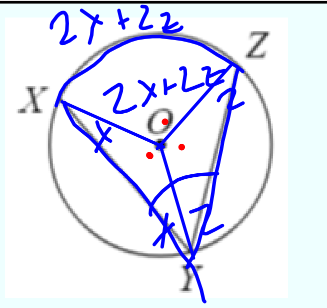
65

(c) Use your answer to the previous part to find $\angle ABC$.

(d) Redo the problem with $\overset{\frown}{BC} = 64^\circ$.

(e) Make a guess about how we can figure out $\angle B$ from the arcs without going through all the steps above.

12.3



360 − (180−2x + 180−2z)

~~360~~ − 360 + 2x + 2z

roblem 12.1, we showed that if an angle is inscribed in a 180° arc (a semicircle), the measure of th

$)^\circ/2 = 90^\circ$. In this problem we will prove that when $\angle XYZ$ is inscribed in $\overset{\frown}{XZ}$ as shown in the diag

he center of the circle, inside $\angle XYZ$), then $\angle Y = \overset{\frown}{XZ}/2$.

Draw $\overline{OX}$, $\overline{OY}$, and $\overline{OZ}$. Let $\angle OXY = x$ and $\angle OZY = z$. Find $\angle XYZ$ in terms of x and z. x+z

Find $\angle XOY$, $\angle YOZ$, and $\angle XOZ$ in terms of x and z. 2x+2z

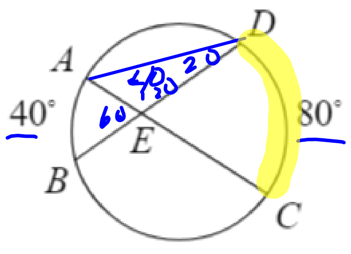
180−2x 180−2z

Find $\overset{\frown}{XZ}$ in terms of x and z, then show that $\angle XYZ = \overset{\frown}{XZ}/2$.

12.7

Secant: A line that intersects a circle in two points.

Secants inside $\frac{x+y}{2}$

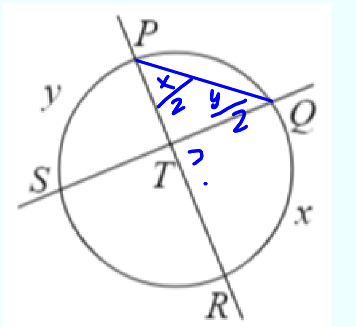


Our goal in this problem is to find $\angle AEB$ given that $\overset{\frown}{AB} = 40°$ and $\overset{\frown}{DC} = 80°$. 60°

(a) Draw $\overline{AD}$ to create inscribed angles. Find $\angle DAE$ and $\angle ADE$. 40° 20°

(b) Find $\angle AED$ and $\angle AEB$. 120 60

(c) How is $\angle AEB$ related to $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$? $\frac{\overset{\frown}{AB} + \overset{\frown}{CD}}{2}$



In this problem we will find a relationship between an angle inside a circle and the arcs the angle intercepts. In the diagram, let $\overset{\frown}{QR} = x$ and $\overset{\frown}{PS} = y$.

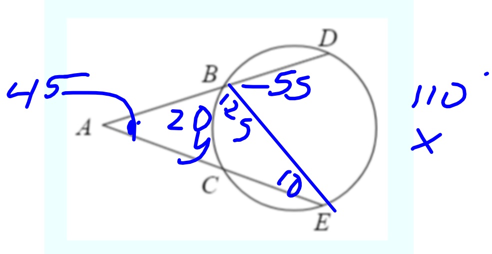
(a) Draw $\overline{PQ}$ and find $\angle PQT$ and $\angle QPT$ in terms of x and/or y.

(b) Find, with proof, $\angle QTR$ in terms of x and y.

$\frac{x+y}{2}$

$\frac{x}{2} + \frac{y}{2} = \angle QTR$ Ext. angle Thm

12.9



In this problem we will find $\angle DAE$ given $\overset{\frown}{DE} = 110^\circ$ and $\overset{\frown}{BC} = 20^\circ$.

(a) Draw $\overline{BE}$ to create inscribed angles. Find $\angle BEC$ and $\angle DBE$.

(b) Find $\angle DAE$. 45°

Secants outside

(c) Can you find a general relationship that must hold among $\angle A$, $\overset{\frown}{BC}$, and $\overset{\frown}{DE}$? (In other words, what if we replace 110° and 20° with x and y?)

$\frac{x-y}{2}$

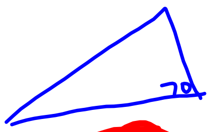
12.12

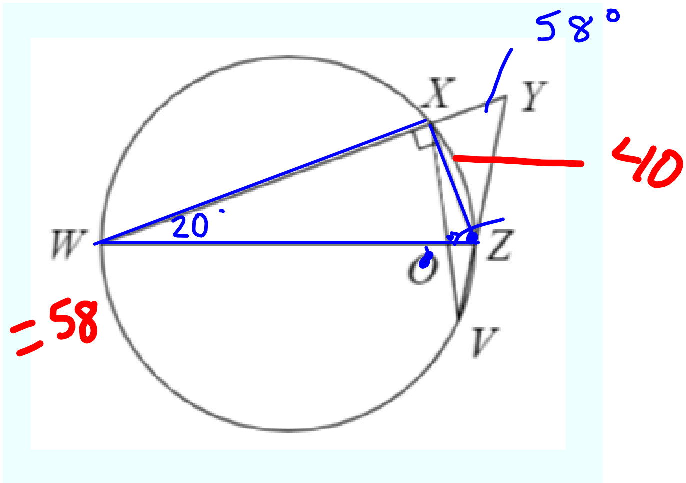
Secants:

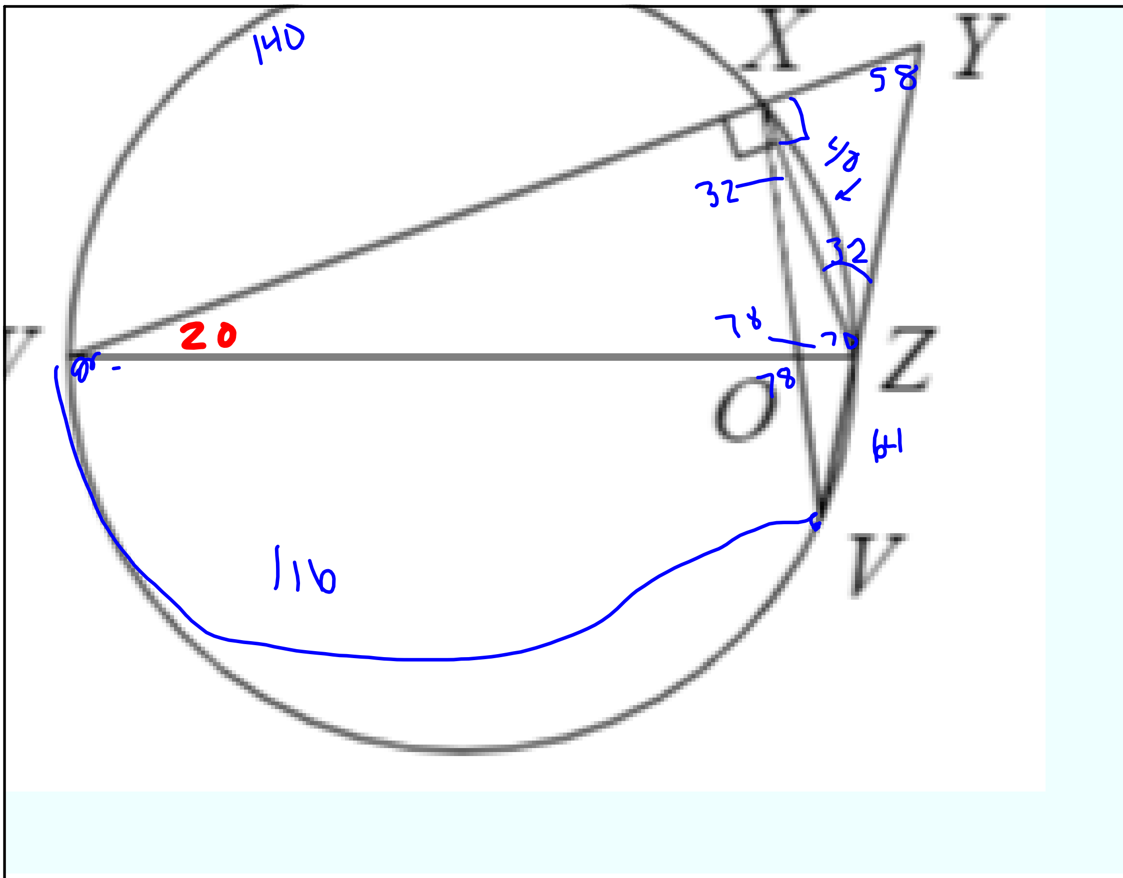
In the figure, $\angle Y = 58^\circ$, $\angle W = 20^\circ$, and $\overline{XZ} \perp \overline{WX}$.

(a) Find $\angle WOV$.

(b) Find $\angle ZXV$.



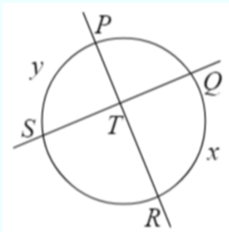




Important:

The measure of the angle formed by two intersecting chords is the average of the measures of the arcs intersected by the chords. For example, in the diagram at right, we have

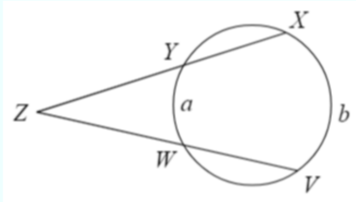
$$\angle PTS = \angle QTR = \frac{\overparen{PS} + \overparen{QR}}{2} = \frac{x+y}{2}.$$



Important:

Two secants that meet at a point outside a circle form an angle equal to half the difference of the arcs they intercept. For example, in the diagram we have

$$\angle Z = \frac{b-a}{2}.$$



Tangent line: touches a circle at one point.

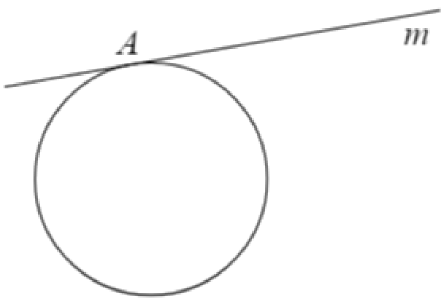
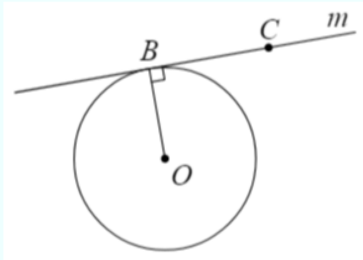


Figure 12.2: A Tangent Line

Important:

A line tangent to a circle is perpendicular to the radius drawn to the point of tangency. Conversely, a line drawn through a point on a circle that is perpendicular to the radius drawn to that point must be tangent to the circle. For example, for the diagram at right, we can write:



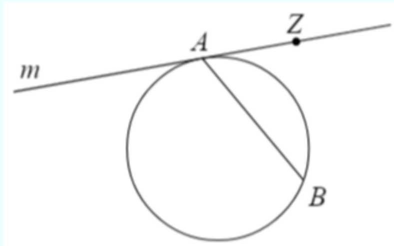
$$\overleftrightarrow{BC} \text{ is perpendicular to radius } \overline{OB} \text{ iff } \overleftrightarrow{BC} \text{ is tangent to circle } O.$$

(Recall that iff is an abbreviation for "if and only if".)

Proof is in your book page 318

Important:

An angle formed by a tangent and a chord with the point of tangency as an endpoint equals one-half the arc intercepted by the angle. For example, in the figure at right, line m is tangent to the circle at A, so

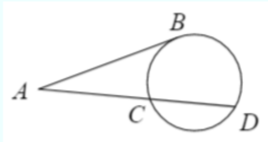


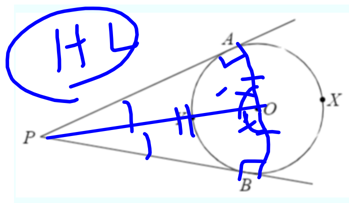
$$\angle ZAB = \frac{\widehat{AB}}{2}.$$

An angle between a tangent and a chord is also often referred to as an **inscribed angle**. Yes, this is the same name we give to an angle between two chords that share an endpoint; the measures of both types of angles equal half the measures of the arcs they cut off.

Important: The angle formed by a tangent and a secant is half the difference of the intercepted arcs. For example,

$$\angle BAD = \frac{\widehat{BD} - \widehat{BC}}{2}.$$





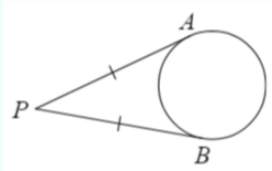
In this problem we investigate the relationship between the lengths of two tangent segments to a circle from the same point, as well as how to find the angle between them. $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent to the circle, which has center O.

(a) Prove that $\triangle POB \cong \triangle POA$.

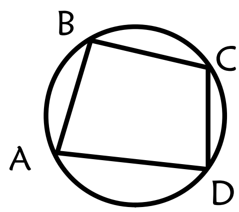
(b) Prove $\angle P = \dfrac{\widehat{AXB} - \widehat{AYB}}{2}$.

The information about the angle between the tangents is unsurprising, as it's essentially the same as the angle between two secants. However, the information about the two tangents from the same point is new, and is one of the most useful tangent facts:

Important: As shown in the diagram at right, we can draw two tangent segments to a circle from a point outside the circle. These tangents are always equal to each other in length.



Cyclic quadrilateral: quadrilateral inscribed in a circle where the circle passes through all of the vertices of the quadrilateral.



Prove that angle A + angle C = 180°

Q.E.D.

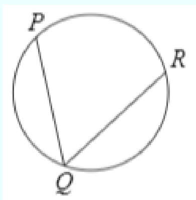
Summary:

Definition: An angle formed by two chords of a circle is **inscribed** in the angle it cuts off.

Important:

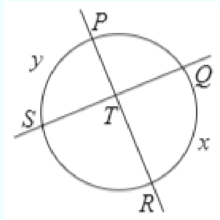
- Any angle inscribed in a semicircle is a right angle.
- An inscribed angle equals $1/2$ the measure of the arc it intercepts. For example,

$$\angle PQR = \frac{\overparen{PR}}{2}.$$



- Any two angles that are inscribed in the same arc are equal.

Important: The measure of the angle formed by two intersecting chords is the average of the measures of the arcs intersected by the chords. For example, in the diagram at right, we have



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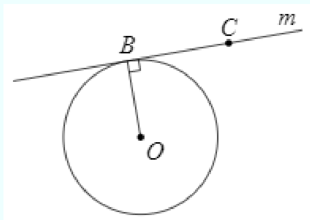

$$\overleftrightarrow{BC} \text{ is perpendicular to radius } \overline{OB} \text{ iff } \overleftrightarrow{BC} \text{ is tangent to circle } O.$$

(Recall that iff is an abbreviation for "if and only if".)

Important: ❗

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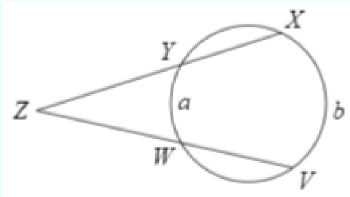

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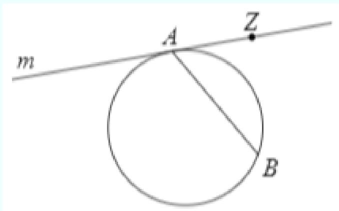

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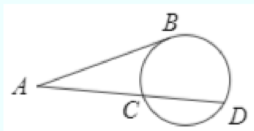

Important:

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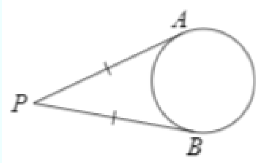


Important:

A **cyclic quadrilateral** is a quadrilateral that can be inscribed in a circle. The opposite angles of any cyclic quadrilateral sum to 180°.

Important:

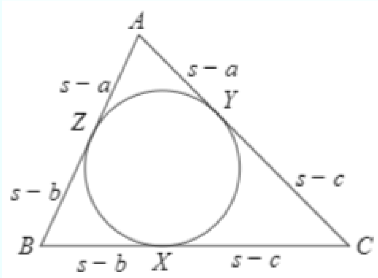
If the length of a median of a triangle is half the length of the side to which it is drawn, the triangle must be a right triangle. Moreover, the side to which this median is drawn is the hypotenuse of the right triangle.

Important:

The lengths from the vertices of $\triangle ABC$ to the points of tangency of its incircle are given as follows:

$$\begin{aligned} AZ &= AY &= s-a \\ BZ &= BX &= s-b \\ CX &= CY &= s-c \end{aligned}$$

where $AB = c$, $AC = b$, and $BC = a$, and the semiperimeter of $\triangle ABC$ is s.



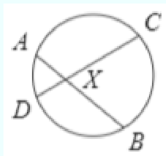
Important:

The length of the inradius of a right triangle with hypotenuse of length c and legs of lengths a and b is

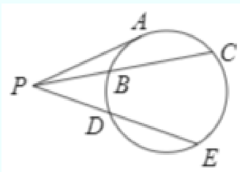
$$\frac{a+b-c}{2}.$$

Important:

Suppose a line through a point P intersects a circle in two points, U and V. The **Power of a Point Theorem** states that for all such lines, the product $(PU)(PV)$ is constant. We call this product the **power** of point P. For example, in the figure at right, applying Power of a Point to X with respect to the circle shown gives



$$(XA)(XB) = (XC)(XD).$$



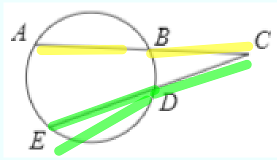
In the figure at left, the power of point P with respect to the circle gives us

$$PA^2 = (PB)(PC) = (PD)(PE).$$

We think of Power of a Point whenever we have a problem involving intersecting segments and a circle.

WARNING!! ⊘

A very common mistake in applying Power of a Point is to write $(BC)(BA) = (DC)(DE)$ when faced with the figure at right . This alleged equality is **not** what Power of a Point tells us! Note that whenever we write a Power of a Point relationship, the same point must appear in *all* of the segments in the equation, as point C does when we write the correct relationship for the secants in the figure above:

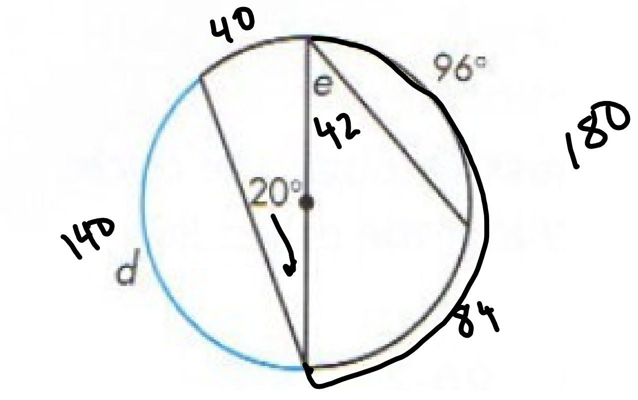


$$(CD)(CE) = (CB)(CA).$$

Let's do some problems.

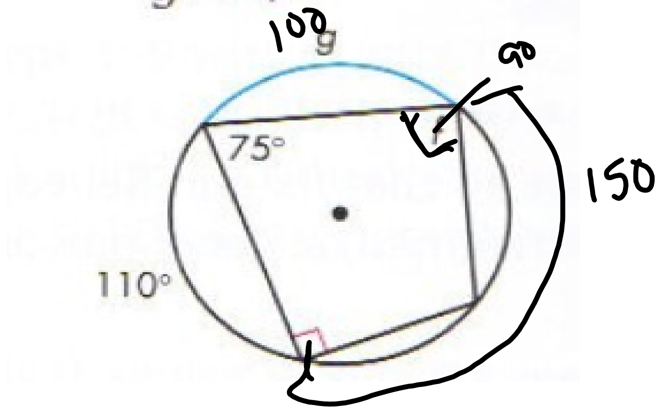
4. $d =$ -?-

$e =$ -?-

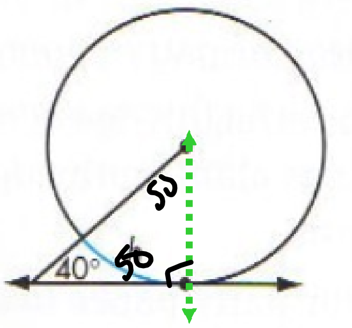


5. $f =$ -?-

$g =$ -?-

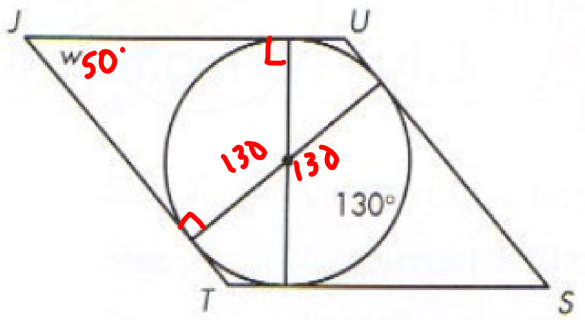


6.* h = -?-

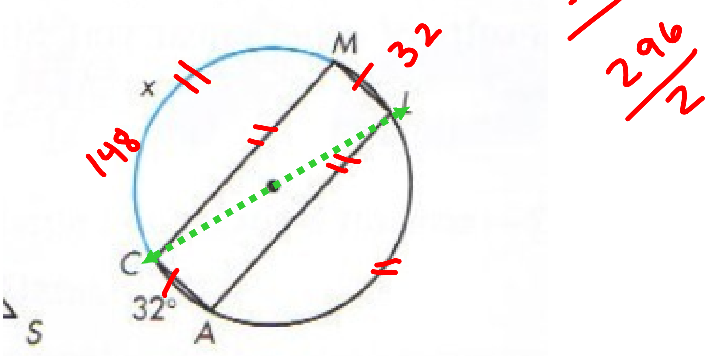


7. *JUST* is a rhombus. 8

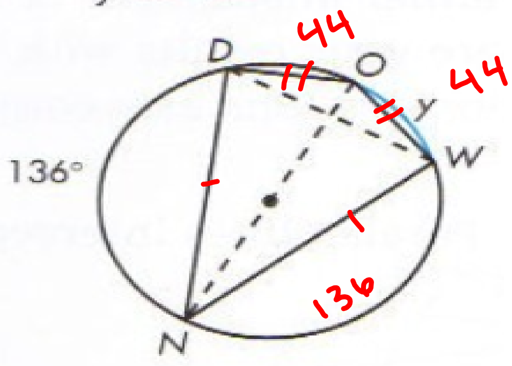
w = -?-



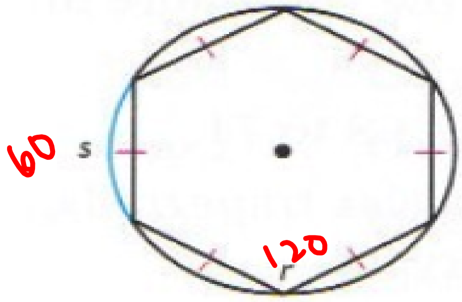
8. *CALM* is a rectangle.
$x = -?-$



9. *DOWN* is a kite.
$y = -?-$



13. $r = $ –?–
$s = $ –?–



Next Steps:

Draw an Altitude from B and show AB > AC.

- Draw an Altitude from B and show AB > AC.

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