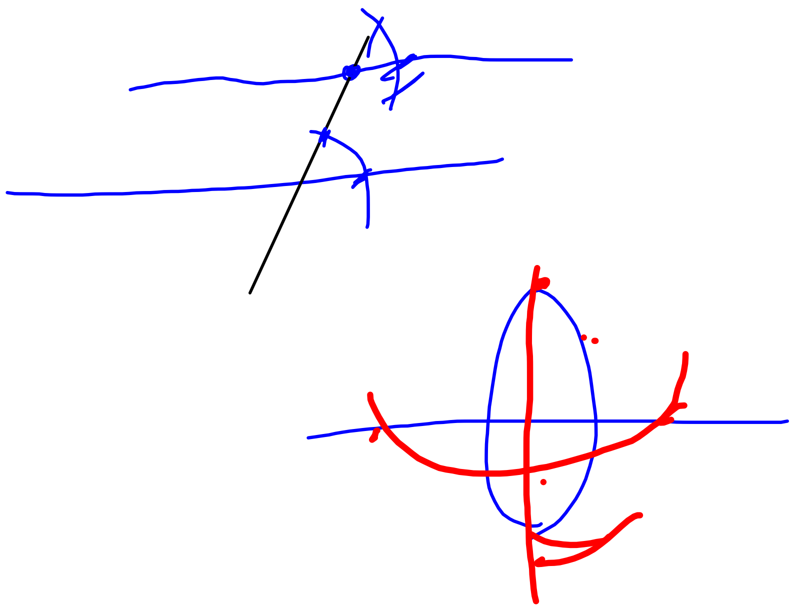
UMTYMP Geometry Day 12

Chapter 14 3D Geometry

Chapter 15 Curved Surfaces

Find your assigned seats today!

Then work on your big circle problem from last week.



2. The following information is true about circle O:

$\overline{AG}$ and $\overline{HL}$ are diameters

$\overline{JK}$ and $\overline{DI}$ are tangents

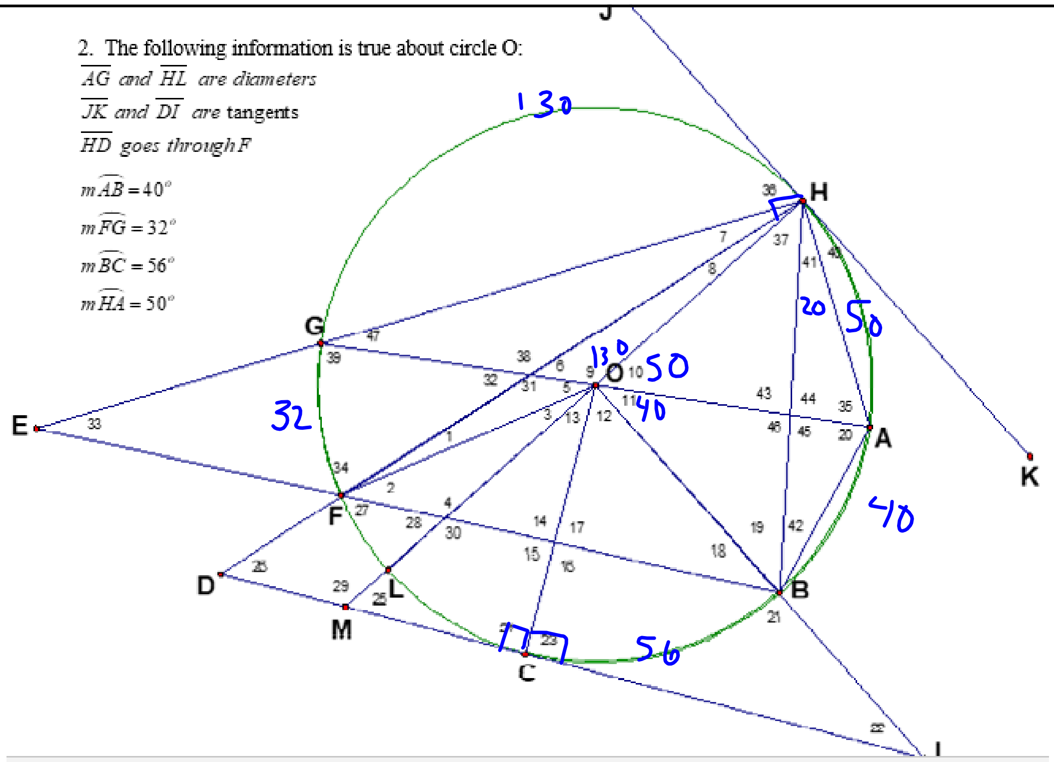
$\overline{HD}$ goes through F

$m\widehat{AB} = 40^\circ$

$m\widehat{FG} = 32^\circ$

$m\widehat{BC} = 56^\circ$

$m\widehat{HA} = 50^\circ$



So far in this book we've confined ourselves to zero dimensions (points), one dimension (lines and line segments), or two dimensions (basically everything else in the book). However, we all know the world around us seems three-dimensional — that in addition to left and right, and forwards and backwards, there's up and down.

A plane is a "flat" two dimensional surface that extends forever.

Collinear points: two or more points on the same line

Coplanar points:

on a plane

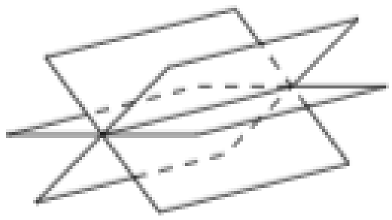
Given two points, how many different planes pass through the two points?

Given three points, is it always possible to find a plane that passes through all three points?

Given any four points, is it always possible to find a plane that passes through all four points?

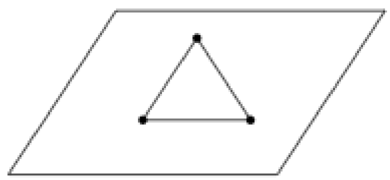
Given two points, how many different planes pass through the two points?

If we only have two points, we can find a line through them, but there are infinitely may planes that can pass through that line (and therefore the two initial points)



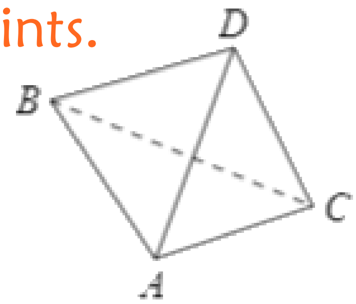
Given three points, is it always possible to find a plane that passes through all three points?

With three (or more) collinear points, we can still find infinitely many planes that pass through all three (since they are on a line...) but if the points are NOT collinear, then we have a triangle. The plane of that triangle is unique.



Given any four points, is it always possible to find a plane that passes through all four points?

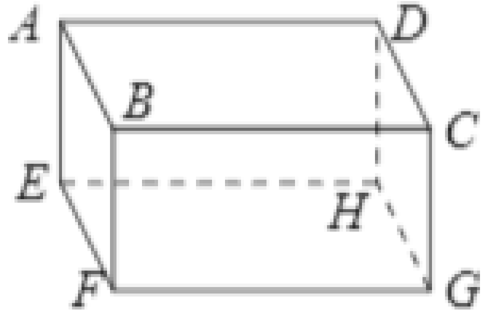
While any three points are coplanar, it is easy to find a set of four points that are noncoplanar. Three vertices of a triangle and a point NOT in the plane of the triangle give us an example in which no plane passes through all four points.



When two lines intersect, their intersection is a point. What happens when two planes intersect?

Is it possible for planes to never intersect?

Can there be a line and a plane that never intersect?



Remember that two lines are // if they are in the same plane but do not intersect. Can two lines <u>in space</u> be such that they are not // but do not intersect?

We call these skew lines.

Given any two intersecting lines, is there always a plane that contains both lines?

Prisms: Two congruent, parallel polygon bases with quadrilateral faces. (AKA a polyhedron)

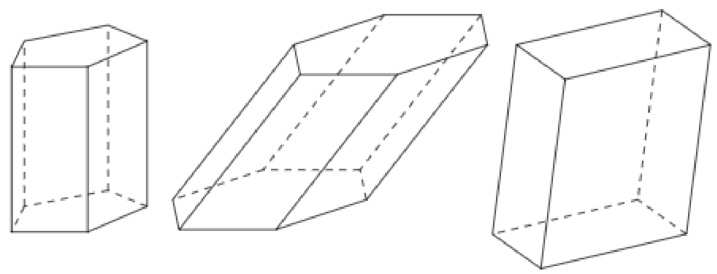


Figure 14.1: Some Prisms

Figure 14.1 shows some prisms. The first is a right regular pentagonal prism because the base is a regular pentagon and the segments connecting the bases are perpendicular to the bases. The second is a hexagonal prism because the bases are hexagons (note that we don't call this one a 'right hexagonal prism'). The third has parallelograms as its bases, as well as all its sides. Such a prism is given the special name parallelepiped. As one last bit of vocabulary, if there's a 'regular' in the description of a prism, it means the base is a regular polygon.

Perimeter: distance around

Area: Space inside 2D

Volume: Space inside 3D

Surface Area: Area of all of the surfaces

Lateral Surface Area: Area of all of the faces (not bases!)

Important: The volume of a prism equals the area of a base times the distance between the bases (i.e. the height).

$b \cdot h$

$V_{prism} = \underbrace{B}_{\text{Area of base}} \cdot h$

At your tables, work on these questions.

Problem 14.4

$ABCDEFGH$ shown is a rectangular prism with $AB = 3$, $BC = 4$, and $AE = 5$.

(a) Find BF and EH. 5 4

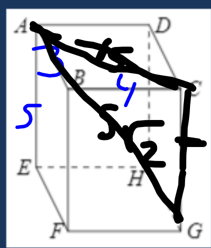
(b) Find the volume of $ABCDEFGH$. $(3 \cdot 4) \cdot 5 = \cancel{12}$ units^3 60 units^3

(c) Find the total surface area of $ABCDEFGH$.

(d) Find AC.

(e) What kind of triangle is $\triangle ACG$?

(f) Find AG.

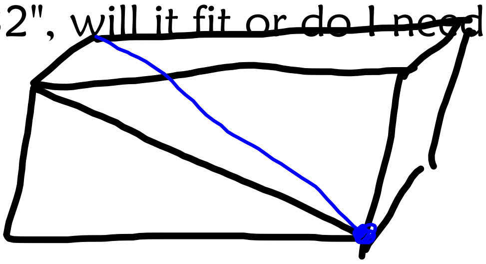


T B	$2 \cdot 3 \cdot 4 = 24$		
L R	$2 \cdot 5 \cdot 3 = 30$	$= 94$ units^2	
FB	$2 \cdot 4 \cdot 5 = 40$		

My son is on a winter travel baseball team. He needs to pack his bats in his suitcase. If his suitcase is 12" deep, 18" wide and 26" tall, what is the longest bat that can fit?

$$18^2 + 26^2 + 12^2 = d^2$$

If his bat is 32", will it fit or do I need to get him a new suitcase?



$$324 + 676 + 144 = \sqrt{1144}$$

33.82"

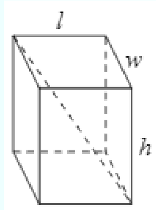
We call this a "space diagonal." Why does it work?

Important:

The three dimensions of a rectangular prism are commonly called the length, l, the width, w, and the height, h. For such a prism, we have:

$B \cdot h$

$$\begin{aligned} \text{Volume} &= lwh \\ \text{Surface area} &= 2(lw + wh + lh) \\ \text{Space Diagonal} &= \sqrt{l^2 + w^2 + h^2} \end{aligned}$$

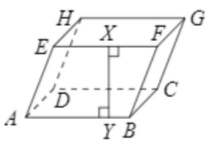


Important:

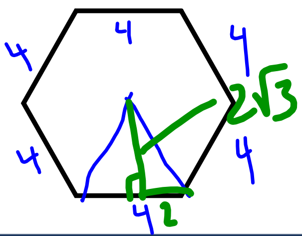
A cube with side length s has:

$$\begin{aligned} \text{Volume} &= s^3 \\ \text{Surface area} &= 6s^2 \\ \text{Space Diagonal} &= s\sqrt{3} \end{aligned}$$

Be careful...



Notice that by 'distance between the bases', we do not necessarily mean the length of the edges connecting corresponding points on the bases. If the prism is not a right prism, then the height is the length of a segment from one base to the other that is perpendicular to both bases. For example, the height between bases $ABCD$ and $EFGH$ of the prism at left is XY, not FB.



At your tables, work on these questions.

$\frac{1}{2}asn$ or $\frac{1}{2}a \cdot p = A_{hexagon}$

Problem 14.7

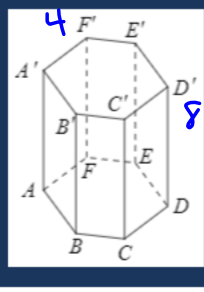
The right regular hexagonal prism shown has sides of length 4 on base $ABCDEF$ and a height of 8 units.

(a) Find the lateral surface area of the prism. sides only

(b) Find the total surface area of the prism.

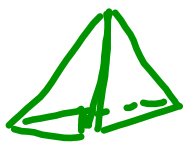
(c) Find the volume of the prism.

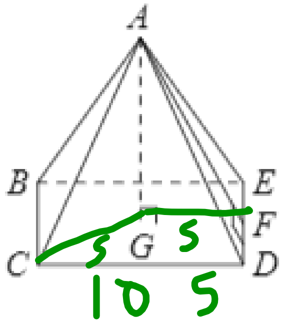
(d) Find AB', AC', and AD'.



Pyramids: Polygon base with triangular faces that meet at a point. (apex)

As with prisms, a pyramid is ‘regular’ if its base is a regular polygon. A regular pyramid is ‘right’ if the center of the base is the foot of the altitude from the apex to the base (i.e., the apex is directly ‘over’ the center of the base).





Usually when we speak of a pyramid, we mean a right pyramid. The height of a pyramid is the distance from the apex to the base. For right regular pyramids, we also define a slant height, which is the distance from the apex to a side of the base. For example, at left, $\overline{AG}$ is the height and $\overline{AF}$ is the slant height of the pyramid.

Important: The volume of a pyramid is one-third the product of the pyramid's height and the area of the pyramid's base.

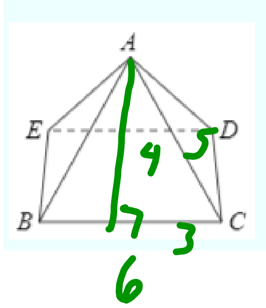
$$\frac{1}{3}\cdot B\cdot h$$

Problem 14.8

$ABCDE$ is a square right pyramid such that $BC = 6$ and $AC = 5$. Find the total surface area and the volume of the pyramid.

$A_{square} + 4\left(\frac{1}{2}(6)4\right)$

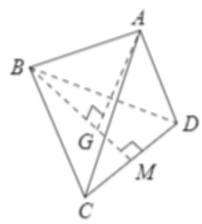
$36 + 48 =$



Problem 14.10

A triangular pyramid is more commonly called a **tetrahedron**. A **regular tetrahedron** is a tetrahedron whose edges all have the same length. Find the volume of a regular tetrahedron that has sides of length 6.

Solution for Problem 14.10: To find the volume, we need an altitude, so we draw altitude $\overline{AG}$ from A to $\triangle BCD$. Since $\overline{AG}$ is perpendicular to plane BCD, triangles $\triangle AGB$, $\triangle AGC$, and $\triangle AGD$ are all right triangles. Because $AB = AC = AD$ and AG is obviously the same in all three triangles, we have $\triangle AGB \cong \triangle AGC \cong \triangle AGD$ by HL Congruence. Therefore, $BG = CG = DG$, which means that G is the circumcenter of $\triangle BCD$ because it is equidistant from the vertices of $\triangle BCD$. Since $\triangle BCD$ is an equilateral triangle, G is also the centroid of $\triangle BCD$.



We can build more right triangles by continuing $\overline{BG}$ to M. Since $\triangle BCD$ is equilateral, $\overline{BM}$ is a median and an altitude. Therefore, $DM = DC/2 = 3$ and $BM = 3\sqrt{3}$ (from 30-60-90 $\triangle BMD$). Since the centroid of a triangle divides its medians in a $2:1$ ratio, we have $BG = (2/3)BM = 2\sqrt{3}$. Finally, we can find AG from right triangle $\triangle AGB$:

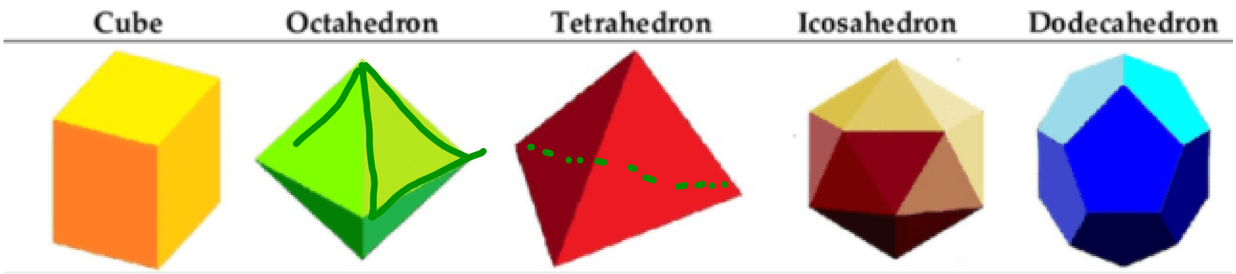
$$AG = \sqrt{AB^2 - BG^2} = \sqrt{36 - 12} = 2\sqrt{6}.$$

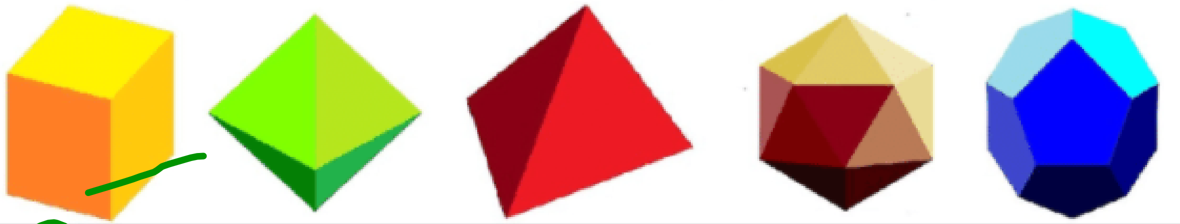
Since the area of $\triangle BCD$ is $(DC)(BM)/2 = 9\sqrt{3}$, our volume is $([BCD])(AG)/3 = 18\sqrt{2}$.

Similarly, we can show that the volume of a regular tetrahedron with edge length s is $s^3\sqrt{2}/12$. □

Platonic Solids:

Name	Face Type	# of Faces	# of Edges	# of Vertices
Tetrahedron		4	6	4
Cube		6	12	8
Octahedron		8	12	
Dodecahedron		12		
Icosahedron		20		





Cube	Octahedron	Tetrahedron	Icosahedron	Dodecahedron
6 faces 8 vertices 12 edges	8 faces 6 vertices 12 edges	4 faces 4 vertices 6 edges	20 faces 12 vertices 30 edges	12 faces 20 vertices 30 edges

Is there a connection between the three?

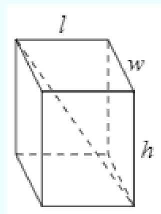
$F + V - 2 = E$

Chapter 14 Summary:

Definitions: The **volume** of a three dimensional figure is a measure of the space inside the figure. The **total surface area** of a figure is the total area of all the surfaces that form a boundaries of the solid. The **lateral surface area** is the total area of all the surfaces that are not considered 'bases'.

Definitions: A **polyhedron** is a solid figure with polygons as its boundaries. A **prism** has two congruent parallel faces as **bases** and all remaining faces (called **sides**) are parallelograms. In a **right prism** all of these side faces are rectangles. The bases are used to describe the prism, as in 'right rectangular prism' (shown below) or 'hexagonal prism'.

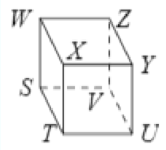
Important: The three dimensions of a right rectangular prism are commonly called the length, l, the width, w, and the height, h. For such a prism, we have:



$$\begin{aligned} \text{Volume} &= lwh \\ \text{Surface area} &= 2(lw + wh + lh) \\ \text{Space Diagonal} &= \sqrt{l^2 + w^2 + h^2} \end{aligned}$$

The volume of a prism equals the area of the base times the distance between the bases (i.e. the height).

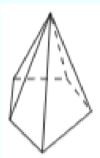
Definition: A **cube** is a special right rectangular prism in which all the edge lengths are the same (i.e., its base is a square and its height has the same length as a side of the base).



Important: A cube with side length s has:

$$\begin{aligned} \text{Volume} &= s^3 \\ \text{Surface area} &= 6s^2 \\ \text{Space Diagonal} &= s\sqrt{3} \end{aligned}$$

Definitions: If we connect all the vertices of a polygon to a point that is not in the same plane as the polygon, we form a **pyramid**. This point is called the **apex** of the pyramid and the polygon is the pyramid's **base**. As we can see at right, the non-base faces of a pyramid are all triangles. The lateral surface area of a pyramid is the sum of the areas of these triangles. A **tetrahedron** is a pyramid with a triangular base.



The **height** of a pyramid is the distance from the apex to the base. If the base is a regular polygon, the pyramid is a **regular pyramid**. For regular pyramids, we also define a **slant height**, which is the distance from the apex to a side of the base.

Important:

- The volume of a pyramid is one-third the product of the pyramid's height and the area of the pyramid's base.
- The lateral surface area of a regular pyramid equals one-half the product of the slant height and the perimeter of the pyramid's base.

Definitions: A **regular polyhedron** is a polyhedron whose faces are all congruent regular polygons.

Important: There are five regular polyhedra, which are described below.

Name	Face Type	# Faces	# Edges	# Vertices
Tetrahedron	Triangle	4	6	4
Cube	Square	6	12	8
Octahedron	Triangle	8	12	6
Dodecahedron	Pentagon	12	30	20
Icosahedron	Triangle	20	30	12

The definition of a prism was "parallel polygon bases with quad re~~ctang~~ular faces." What happens when the base ISN'T a polygon?

We have cylinders!

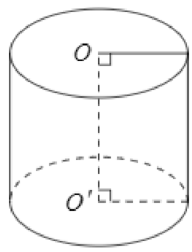


Figure 15.1: A Right Circular Cylinder

The radius of the base is the radius of the cylinder. OO' is the height/axis of the cylinder.

In your groups...

Problem 15.1

(a) Suppose we take a cross-section of a cylinder that is perpendicular to the axis of the cylinder. What shape is this cross-section?

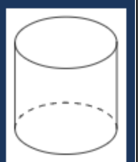
(b) What is the shape of a cross-section of a cylinder that contains the axis of the cylinder?

(c) What is the shape of a cross-section of a cylinder that is parallel to the axis of the cylinder?

In your groups...

Problem 15.2

The figure shows a right circular cylinder (a.k.a. a cylinder) with radius 3 and height 5.



(a) Find the volume of the cylinder.

(b) What is the lateral surface area of our cylinder?

(c) What is the total surface area of our cylinder?

(d) Find formulas for the volume, lateral surface area, and total surface area of a cylinder with radius r and height h.

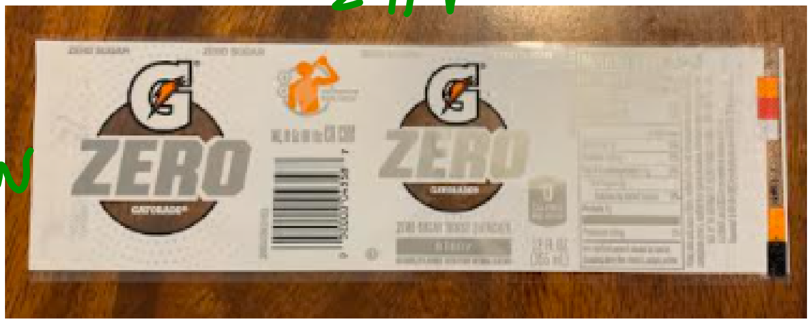
$SA = 2\pi r^2 + 2\pi r h$

$\pi r^2 \cdot h$

$B \cdot h$

$2\pi r$

w



Important: A cylinder with height h and radius r has:

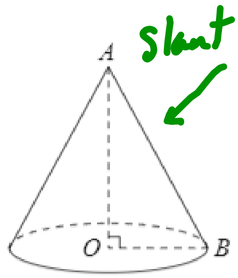
$$\begin{aligned} \text{Volume} &= \pi r^2 h \\ \text{Lateral Surface Area} &= 2\pi r h \\ \text{Total Surface Area} &= 2\pi r h + 2\pi r^2 \end{aligned}$$

Don't memorize these formulas! If you take the time to understand them, they'll always be obvious to you.

Concept: Problems involving the curved surface of a cylinder can often be untangled by 'unrolling' the curved surface into a rectangle.

A pyramid with a circular base is a ... cone!

The figure on the right shows a right circular cone. The point A at the tip of the cone is the vertex of the cone and the distance from the vertex to the base is the height. The line connecting the vertex to the center of the base is the axis of the cone. The radius of the base is considered the radius of the cone, and for right circular cones, the distance from the vertex to a point on the circumference of the base is the slant height



Important: The volume of a circular cone with height h and radius r is

$$\text{Volume} = \frac{1}{3}\pi r^2 h.$$

In your groups...

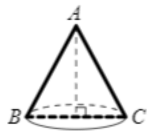
Problem 15.4

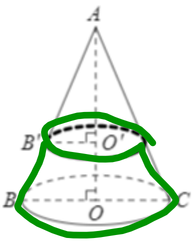
(a) What is the shape of a cross-section of a cone that contains the axis of the cone?

(b) What is the shape of a cross-section of a cone that is perpendicular to the axis of the cone?

Solution for Problem 15.4:

(a) A cross-section of a cone that contains the cone's axis consists of two segments of equal length connecting the vertex to two points on the circumference of the base of the cone, as well as the segment connecting these two points along the base of the cone. So, our cross-section is an isosceles triangle, such as $\triangle ABC$ in the diagram at right.





Intuitively, it seems clear that a cross-section of a cone perpendicular to the axis of the cone is a circle. Suppose the plane of our cross-section meets the axis at O'. To prove our cross-section is a circle, we must show that every point where our plane hits the curved surface of the cone is equidistant from O'. Consider point B', the intersection of $\overline{AB}$ and our cross-section plane, as shown. Since $\overline{B'O'}$ and $\overline{BO}$ are each perpendicular to $\overline{AO}$, we have $\overline{B'O'} \parallel \overline{BO}$. Therefore, $\triangle AO'B' \sim \triangle AOB$. Since $\triangle AO'B' \sim \triangle AOB$, we have $B'O'/BO = AO'/AO$. Therefore, we find $B'O' = (AO'/AO)(BO)$. Since BO is just the radius of the cone and AO is the cone's height, we have $B'O' = (r/h)(AO')$. Similarly, we can show that all points of the cross-section are $(r/h)(AO')$ away from O'. Since AO' is fixed, all the points of our cross-section are the same distance from O'. Therefore, the cross-section must be a circle. (Make sure you see why every point on this circle must be in the cross-section.)

frustrum

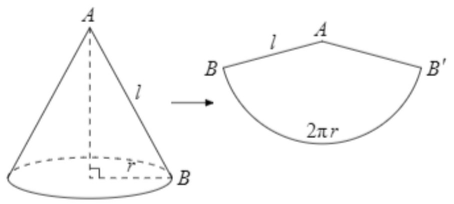
In your groups...

Problem 15.5

Find a formula for the lateral surface area of a right circular cone with base radius r and slant height l.

☆

olution for Problem 15.5: Since cutting and unrolling the curved surface was so uccessful in finding the lateral surface area of a cylinder, we try it with a cone as ell. We cut the curved surface of the cone along $\overline{AB}$, where A is the vertex and B is a point on the circumference of the base.



ince every point on the circumference of the cone's base is the same distance rom the cone's vertex (the slant height), when we unroll the curved surface, these oints will still all be the same distance from the vertex. Hence, our 'unrolled' urface is a sector of a circle as shown at right above. (B and B' coincide when he sector is rolled up to form a cone.)

he radius of this sector is AB, the slant height of the cone. To find the area of the sector, we must determine what portion of a whole ircle the sector is. We know that the length of $\widehat{BB'}$ is just equal to the circumference of the cone's base. The base of the cone has radius , so its circumference is $2\pi r$. Thus, the length of $\widehat{BB'}$ is $2\pi r$. Since a whole circle with radius $AB = l$ has circumference $2\pi l$, our sector s $(2\pi r)/(2\pi l) = r/l$ of a whole circle.

full circle with radius l has area πl^2, so the area of a sector that is r/l of this circle is $(r/l)(\pi l^2) = \pi r l$. Recall from Problem 14.9 that e showed that the lateral surface area of a regular pyramid is half the product of the slant height and the perimeter of the base. The proof e used there wouldn't work for cones, since we don't have triangular faces as the sides of a cone. However, since cones are essentially ust pyramids with circular bases, we expect the formula to work for cones, too. Trying it, we note that the perimeter of the base of a cone s $2\pi r$, so our formula gives us $(1/2)(2\pi r)(l) = \pi r l$ for the lateral surface area. Unsurprisingly, this matches the formula we already roved. □

Important: The lateral surface area of a right circular cone with radius r and slant height l is $\pi r l$.

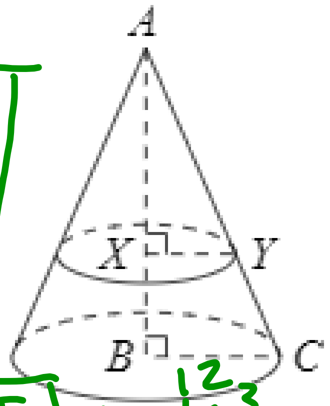
Lateral Surface area $\pi r l$

full S.A. $\pi r^2 + \pi r l$

In your groups...

Problem 15.8

A cone with vertex A, height $AB = 9$, and radius $BC = 12$ is given. The cone is cut in two pieces by a plane perpendicular to $\overline{AB}$ at point X, where $AX = 6$. Find the volume of the two smaller pieces thus formed.



$\frac{1}{3}\pi r^2 \cdot h$

$\frac{1}{3}(\pi)(12)^2 \cdot 9$

$\frac{1}{3} \cdot \pi \cdot 144 \cdot 9$

$\pi \cdot 144 \cdot 3$

9

12

$\boxed{432\pi}$ units^3

exact

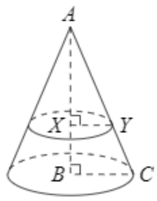
approx

128π

304π

frustrum

Solution for Problem 15.8: We showed in Problem 15.4 that a cross-section of a cone perpendicular to its axis is a circle. So, one of our pieces is itself a cone. The other piece is called a **right circular frustum**. We don't have any tools to deal with a frustum, but we do know how to find the volume of a cone. The original cone has volume $\pi r^2 h/3 = \pi(12^2)(9)/3 = 432\pi$. We have the height of the smaller cone, $AX = 6$, so all we have to do is find the radius.



Since $\overline{XY}$ and $\overline{BC}$ are each perpendicular to $\overline{AB}$, we have $\angle AXY = \angle ABC$ and $\angle XAY = \angle BAC$, so $\triangle AXY \sim \triangle ABC$. (Notice that we are essentially considering the cross-section of the cone that contains $\triangle ABC$ here – three-dimensional problems are often just two-dimensional problems in disguise!) Therefore, $XY/AX = BC/AB$, so

$$XY = (BC/AB)(AX) = (12/9)(6) = 8.$$

Hence, our little cone has volume $\pi(XY^2)(AX)/3 = 128\pi$.

To get the volume of the other piece, we merely subtract the little cone from the big one, which yields $432\pi - 128\pi = 304\pi$.

Notice that

$$\frac{\text{Volume of small cone}}{\text{Volume of large cone}} = \frac{128\pi}{432\pi} = \frac{8}{27} = \left(\frac{2}{3}\right)^3 = \left(\frac{AX}{AB}\right)^3.$$

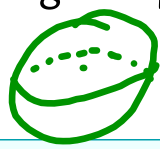
This shouldn't be a surprise, because our cones are similar figures. □

Important: Just as the ratio of the areas of similar two-dimensional figures is the square of the ratio of their corresponding sides, the ratio of the surface areas of similar three-dimensional figures is the square of the ratio of their corresponding side lengths. Moreover, the ratio of the volumes of similar three-dimensional figures is the cube of the ratio of their corresponding side lengths.

$$\frac{m}{n} = \frac{m^2}{n^2} = \frac{m^3}{n^3}$$ ☆

Sides Areas Volume

Just as a circle is the set of all points in a plane that are the same distance from a given point, a sphere is the set of all points in space that are equidistant from a given point.



Important: A sphere of radius r has:

$$\text{Volume} = \frac{4\pi r^3}{3}$$

$$\text{Surface Area} = 4\pi r^2$$

How do you keep these formulas straight? (pun intended! ;-))

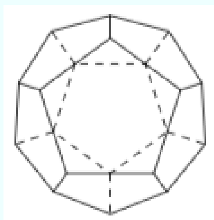
Important: Every cross-section of a sphere is a circle (or a point, when the cross-section plane is tangent to the sphere). The segment connecting the center of the sphere to the center of this circle is perpendicular to the plane of the cross-section.

PLEASE carefully read through the proof of this in question 15.10 in your book.

A cross-section of a sphere that has the center of the sphere as its center is sometimes called a great circle of the sphere.

Extra!

We saw in **here** that squares, hexagons, and triangles are the only regular polygons that will tile the plane. Pentagons, with their quirky 108° angles, simply can't add up to 360°, no matter how many of them get together at a vertex. But, on a sphere, pentagons get their due! We can view the dodecahedron we discovered in Section 14.4 **here** (and shown at right) as a tiling of a sphere with regular pentagons.



Each of the other types of polyhedra can be considered a method of tiling a sphere with regular polygons. Notice that while there is only one way to tile a plane with equilateral triangles, there are three ways to tile a sphere with them!

There's also one well-known example of a tiling that uses both hexagons and pentagons – it's commonly known as a soccer ball. This fabulous structure has been around since before the invention of soccer, too, in the form of \Def{buckminsterfullerene}, C_{60}, a recently discovered form of carbon. Drs. Richard Smalley and Robert Curl received the Nobel Prize in 1996 for that discovery.

The mathematician Johannes Kepler wondered how densely spheres can fill space. The typical stacking you see for oranges at the grocery store fills just 74% of space. Is there a different arrangement that gets more oranges in the same space? It took nearly 400 years before mathematicians Thomas Hales and Samuel Ferguson were able to answer Kepler's question and prove in 1998 what grocers have known all along: there isn't a better way to pack oranges.

14.6

Problem 14.6 Jump to Solution

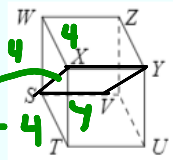
Once again, we consider cube $STUVWXYZ$.

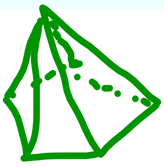
(a) What geometric shape is $SXYV$?

(b) What is the area of $SXYV$ if $SV = 4$?

$16\sqrt{2}$

$4\sqrt{2}$





pentagon

$4 \cdot 4\sqrt{2}$