

# UMTYMP Geometry Day 14

## Chapter 17 Analytic Geometry Part 2

(17.4-17.7)

## Chapter 18 Trigonometry and Right Triangles

If the coordinates of points A and B are  $(x_1, y_1)$  and  $(x_2, y_2)$ , then

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Midpoint} = \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$$

## Proofs with Analytic Geometry:

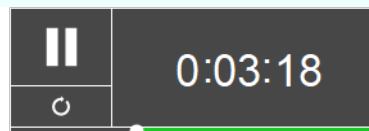
### Example 1:

With a group of 3-4 people near you, prove the midpoint of the hypotenuse of a right triangle is the circumcenter. (5 min)

## Proofs with Analytic Geometry:

Prove the midpoint of the hypotenuse of a right triangle is the circumcenter.

- (a) Let the right triangle be  $\triangle ABC$ , with right angle at  $C$ . Why is it useful to choose  $C$  to be the origin of the Cartesian plane?
- (b) Suppose  $C$  is the origin. Why should we choose the axes of the Cartesian plane such that the legs of  $\triangle ABC$  are along the axes?
- (c) Let  $A$  be  $(a, 0)$  and  $B$  be  $(0, b)$ . What is the midpoint of the hypotenuse?
- (d) Show that the midpoint of the hypotenuse is equidistant from the vertices of the triangle.

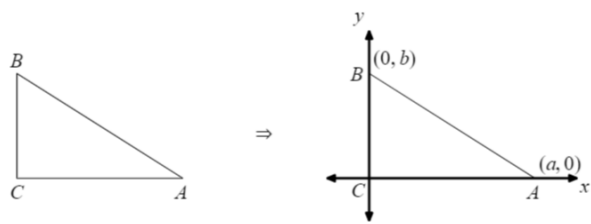


#### WARNING!!

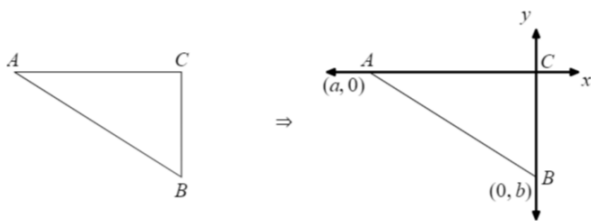


When setting up a proof on the Cartesian plane, we must be very careful. Our proof must address all possible configurations of the problem, so our analytic geometry representation of the problem must cover all possible configurations.

We might start by letting  $A = (a, b)$ ,  $B = (c, d)$ , and  $C = (e, f)$ . But that's six variables! Hopefully we can find a simpler representation of  $\triangle ABC$ . When setting up a geometry problem on the Cartesian plane, we can start with a diagram, then add the coordinate axes. We try to do so in a way that simplifies our problem. We can choose any point to be the origin, so we choose one of the vertices of the right triangle to be our origin.



We choose the vertex of the right angle, point  $C$ , to be the origin because this allows us to choose our axes so that the legs of the right triangle are along the axes. So, we let  $A = (a, 0)$  and  $B = (0, b)$ . Notice that  $a$  and/or  $b$  could be positive or negative. For example, in the case below, both  $a$  and  $b$  are negative.



Since  $C$  is the vertex of the right angle, side  $\overline{AB}$  is the hypotenuse. Let  $M$  be the midpoint of  $\overline{AB}$ , so that

$$M = \left( \frac{a}{2}, \frac{b}{2} \right).$$

To show that  $M$  is the circumcenter of  $\triangle ABC$ , we must show that  $M$  is equidistant from all three vertices. This is a job for the distance formula:

$$AM = \sqrt{\left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\frac{a^2 + b^2}{4}},$$

$$BM = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2} = \sqrt{\frac{a^2 + b^2}{4}},$$

$$CM = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}.$$

We have  $AM = BM = CM$ , so  $M$  is the circumcenter of  $\triangle ABC$ . Notice that every step in our proof is valid even if  $a$  or  $b$  or both are negative.  $\square$

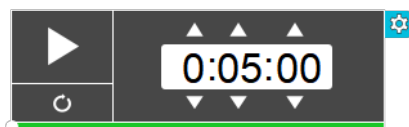
**Important:**

When setting up a geometry problem on the Cartesian plane, choose your origin and your coordinate axes wisely. Typically, we do so in a way that makes the coordinates of important points in the problem as simple as possible.

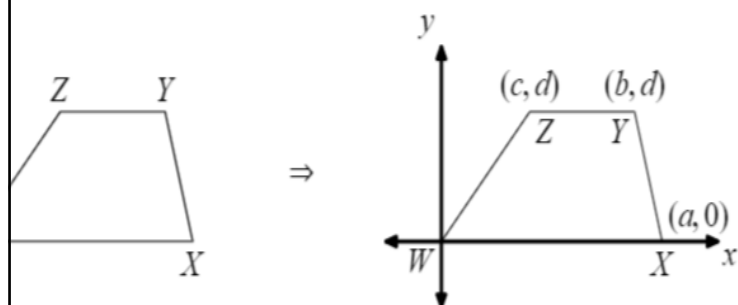
Notice that our proof in Problem 17.14 only applies to right triangles, not to all triangles. The points we chose to represent the vertices,  $(0, 0)$ ,  $(a, 0)$ , and  $(0, b)$ , are always the vertices of a right triangle (when  $a$  and  $b$  are nonzero). We cannot use these three points as our vertices to prove a fact about all triangles, since any triangle with vertices  $(0, 0)$ ,  $(a, 0)$ , and  $(0, b)$  must be a right triangle.

## Example 2:

Prove that the median of a trapezoid is parallel to the bases of the trapezoid, and that the length of the median is the average of the lengths of the bases. (5 min)

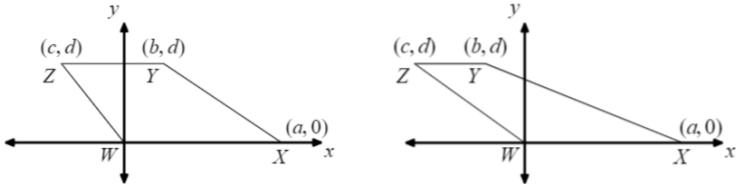


our trapezoid be  $(a, b)$ ,  $(c, d)$ ,  $(e, f)$ , and  $(g, h)$ . Yikes. Eight variables! Maybe we can do better. With a diagram, then add the coordinate axes to set up a proof on the Cartesian plane. Here, we place the bases such that the other base is above the  $x$ -axis. We then choose the leftmost of the two bases so that one vertex is the origin and the other is  $(a, 0)$  with  $a > 0$ . We'll call the origin  $W$  and the second



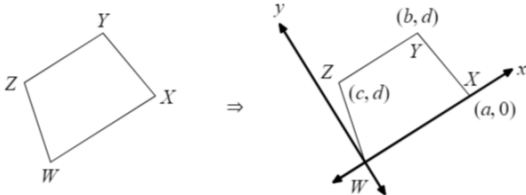
We placed the  $x$ -axis such that the endpoints of one base are on the  $x$ -axis because this forces the endpoints of the other base to have the same  $y$ -coordinate, since the bases are parallel. By choosing the 'lower' base to be  $x$ -axis, we force this common  $y$ -coordinate to be positive. Therefore, the other two vertices of the trapezoid can be represented by  $(b, d)$  and  $(c, d)$ , where  $d > 0$  and  $b > c$ . Finally, we let  $Y$  be  $(b, d)$  and  $Z$  be  $(c, d)$ , so  $\overline{XY}$  and  $\overline{WZ}$  are the legs of the trapezoid. Now, we have represented our trapezoid with only 4 variables, instead of 8.

Notice that every trapezoid can be represented in this way. Because  $b$  and  $c$  can take on negative values, letting the vertices be  $W = (0, 0)$ ,  $X = (a, 0)$ ,  $Y = (b, d)$ , and  $Z = (c, d)$  with  $a > 0$ ,  $d > 0$ , and  $b > c$  will also represent trapezoids in which  $Z$  and/or  $Y$  end up to the left of the  $y$ -axis. Two examples are shown below. In each case, we have  $W$  at the origin,  $X$  to the right of  $W$  on the  $x$ -axis, and  $\overline{YX}$  and  $\overline{ZW}$  as the legs (because  $b > c$ ).



But what about trapezoids in which the bases are not horizontal?

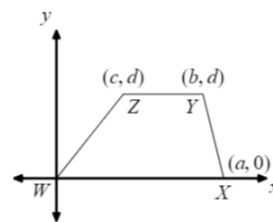
There's no reason we have to make our  $x$ -axis and  $y$ -axis horizontal and vertical! They only have to be perpendicular. For example, consider the trapezoid with 'slanted' bases below. We similarly make our axes 'slanted' so that the  $x$ -axis includes one base.



Now, we're ready for our proof. We'll use the diagram in which  $\overline{YZ}$  is horizontal and both  $Y$  and  $Z$  are to the right of the  $y$ -axis, as shown at right. (However, our proof will be valid for all set-ups we have discussed.) The midpoint of leg  $\overline{WZ}$  is  $(\frac{c}{2}, \frac{d}{2})$  and the midpoint of leg  $\overline{XY}$  is  $(\frac{a+b}{2}, \frac{d}{2})$ . The median connects these two points, so the median is on the line  $y = \frac{d}{2}$ . Therefore, the median is horizontal, so it is parallel to the bases. We can also use the coordinates of the midpoints of the legs to determine that the length of the median is

$$\frac{a+b}{2} - \frac{c}{2} = \frac{a+b-c}{2}.$$

The lengths of the bases are  $WX = a$  and  $YZ = b - c$ , so the average of the lengths of the bases is also  $(a + b - c)/2$ . So, the length of the median equals the average of the lengths of the bases. Notice that every step of this proof is valid for all the other arrangements we showed. We also could have avoided the fractions in our solution by being a little clever about assigning variables to coordinates. Because we are going to have to work with midpoints to get information about the median, we know we'll have to divide expressions by 2. Therefore, we might make our coordinates  $W = (0, 0)$ ,  $X = (2a, 0)$ ,  $Y = (2b, 2d)$ , and  $Z = (2c, 2d)$ . Then, the endpoints of the median are  $(a + b, d)$  and  $(c, d)$ . No fractions! See if you can finish the problem from here.  $\square$

**Concept:**

When we set up a geometry problem involving midpoints on the Cartesian plane, we often use  $2a$ ,  $2b$ ,  $2c$ , etc. for coordinates, rather than just  $a$ ,  $b$ ,  $c$ , etc. This helps us avoid expressions involving fractions when we find midpoints of segments in the problem.

**WARNING!!**

When using analytic geometry for a proof, be extra careful not to overlook special cases.

**Concept:**

Don't make analytic geometry proofs harder than they need to be!

### Example 3:

Prove that if the diagonals of a quadrilateral are perpendicular and bisect each other, the quadrilateral is a rhombus.

(hint...align the center at the origin and diagonals along the axes)

## 17.5 Distance between a point and a line

Find a formula for the distance between the point  $(x_0, y_0)$  and the line  $Ax + By + C = 0$ , where  $A$ ,  $B$ ,  $C$ ,  $x_0$ , and  $y_0$  are all constants.



*Solution for Problem 17.19:* Let  $P$  be the point  $(x_0, y_0)$  and  $k$  be the graph of  $Ax + By + C = 0$ . We could proceed as we did in our solution to Problem 17.18, by finding the slope of  $k$ , then finding the equation of the line through  $P$  that is perpendicular to  $k$ . Then, we find the intersection of  $k$  and this new line, and then. . .

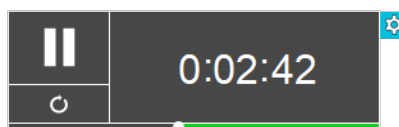
That looks like a lot of work. Before diving into pages of algebra, let's see if we can use some geometric insights to simplify the problem.

**Concept:**



Algebra is not the only tool we have to solve problems about the Cartesian plane. Combining geometric insights with algebra can lead to very nice solutions.

Try this with your groups (5 min)



The distance between the point  $(x_0, y_0)$  and the graph of the equation  $Ax + By + C = 0$  is

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

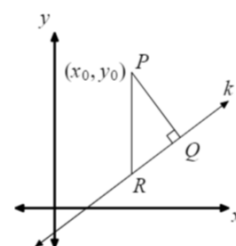
We start by drawing a diagram with  $P$ ,  $k$ , and the perpendicular segment from  $P$  to  $k$ . Let point  $Q$  be the foot of the perpendicular from  $P$  to  $k$ . We seek a length, and we have a right angle, so we think about building right triangles. We build a right triangle by drawing a vertical segment from  $P$  to line  $k$ , meeting  $k$  at point  $R$ , as shown.

Since  $P$  and  $R$  are on the same vertical line, the  $x$ -coordinate of  $R$  is  $x_0$ . Since  $R$  is on  $k$ , its coordinates satisfy the equation  $Ax + By + C = 0$ . We can now use this equation to find the  $y$ -coordinate of  $R$ . Let  $y_R$  be the  $y$ -coordinate of  $R$ , so  $R$  is  $(x_0, y_R)$ . Because  $R$  is on  $k$ , we must have

$$Ax_0 + By_R + C = 0.$$

Solving for  $y_R$  gives us

$$y_R = \frac{-Ax_0 - C}{B}.$$



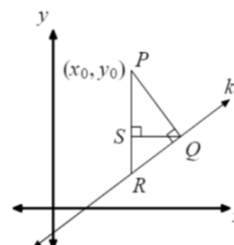
It's still not clear how we can find the coordinates of  $Q$ . Right triangles make us think of similar triangles. Drawing the altitude  $\overline{QS}$  to the hypotenuse of right triangle  $\triangle PQR$  gives us plenty of similar triangles.

But what good are they? We haven't used any information about the equation of our line yet, so we focus on that. When we do so, we see that we can relate lengths of segments in our diagram to the slope of  $k$ . Specifically, we see that the slope of line  $k$  equals  $SR/QS$ . (In our diagram, we assume the slope of  $k$  is positive. Essentially the same approach will work if the slope of  $k$  is negative.) We can also write the slope of  $k$  in terms of  $A$  and  $B$ . Putting the equation  $Ax + By + C = 0$  in slope-intercept form gives

$$y = -\frac{A}{B}x - \frac{C}{B},$$

so the slope of  $k$  is  $-A/B$ . Therefore, we have

$$\frac{SR}{QS} = -\frac{A}{B}.$$



Our similar triangles give us a way to relate  $\triangle QSR$  to  $\triangle PQR$ . We have  $\triangle QSR \sim \triangle PQR$ , so

$$\frac{SR}{QS} = \frac{QR}{PQ}.$$

Combining this with  $SR/QS = -A/B$  gives us  $QR/PQ = -A/B$ . Rearranging this gives  $PQ = (-B/A)(QR)$ .

Unfortunately, it's not so clear how to find  $QR$ . But we can find  $PR$ . Points  $P$  and  $R$  have the same  $x$ -coordinate, so  $PR$  equals the difference in the  $y$ -coordinates of  $P$  and  $R$ :

$$PR = y_0 - y_R = y_0 - \frac{-Ax_0 - C}{B} = \frac{Ax_0 + By_0 + C}{B}.$$

Now, we're close. We have  $PR$ , and we can relate  $PQ$  to  $QR$ . Finally, we use the Pythagorean Theorem to finish. We have  $PR^2 = PQ^2 + QR^2$  and  $QR = -(A/B)PQ$ , so we have

$$\left(\frac{Ax_0 + By_0 + C}{B}\right)^2 = PQ^2 + \left(\frac{A^2}{B^2}\right)PQ^2.$$

Multiplying both sides by  $B^2$  gives

$$(Ax_0 + By_0 + C)^2 = B^2 \cdot PQ^2 + A^2 \cdot PQ^2 = (A^2 + B^2)(PQ^2).$$

Dividing both sides by  $A^2 + B^2$ , then taking the square root of both sides, gives

$$PQ = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

We need the absolute value on the right side because length must be positive.

Our proof does not address the cases in which  $k$  is horizontal or vertical – you'll be asked to tackle these cases as an Exercise. Moreover, our proof assumes that the slope of  $k$  is positive, and that  $P$  is above the line. However, the proofs are essentially the same for other possible configurations in which  $k$  is neither horizontal nor vertical.

Also, our proof doesn't address the possibility that  $P$  is on  $k$ . We can quickly show that our formula works in this case. If  $P = (x_0, y_0)$  is on the graph of  $Ax + By + C = 0$ , then we must have  $Ax_0 + By_0 + C = 0$ . When we substitute  $Ax_0 + By_0 + C = 0$  in our formula, then our formula gives us a distance of 0, which is indeed the correct distance between point  $P$  and line  $k$  when  $P$  is on  $k$ .  $\square$

17.5.3 At how many points does the graph of  $2x-3y=48$  intersect the graph of  $(x-3)^2+(y+2)^2=100$ ?

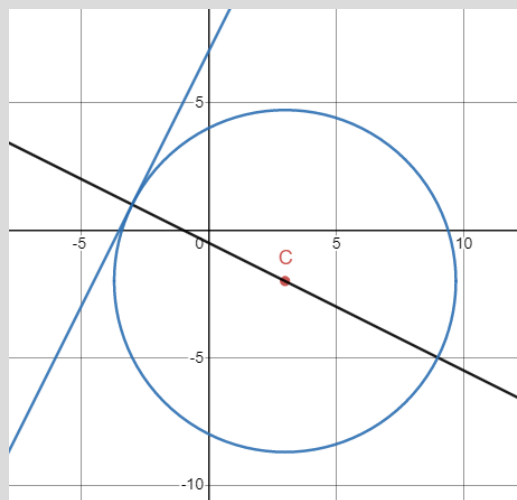
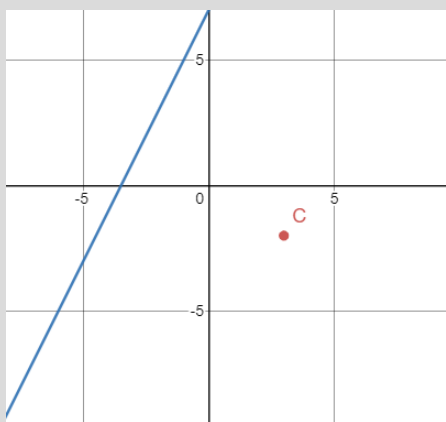
Shift both objects towards the origin the same amount...

$$\begin{aligned}x^2 + y^2 &= 100 \\ 2(x+3) + (y-2) &= 100\end{aligned}$$

Simplify eq for line, then determine using distance formula if the line is within a radius' distance from the origin (yes, just barely).

## 17.6 Advanced Analytic Geometry Problems

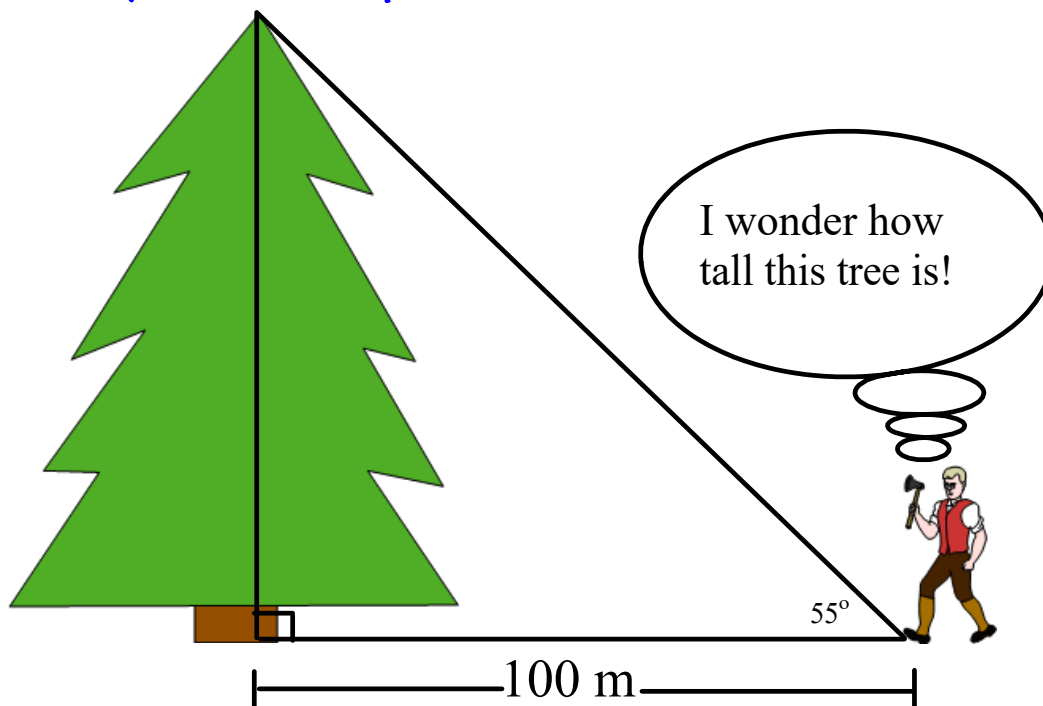
Let point C be (3, -2). Points A and B are on the graph of  $2x - y = -7$  such that triangle ABC is equilateral. Find AB.



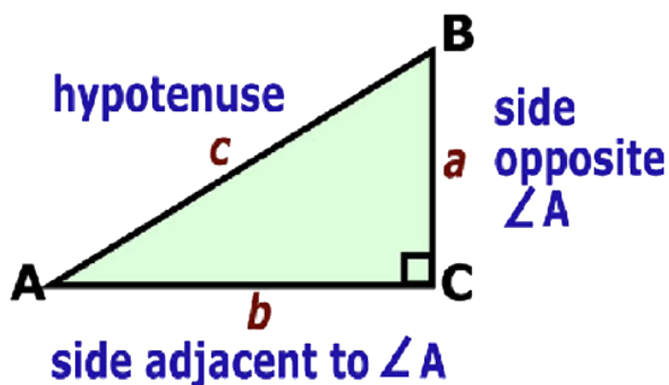
# Right Triangle Trigonometry:

The study of the relationship between the sides and the angles of right triangles.

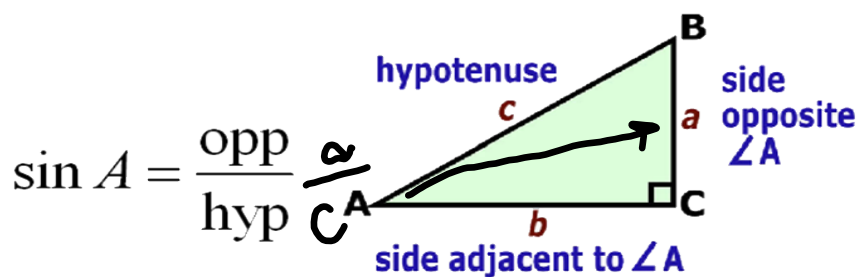
Why is this important?



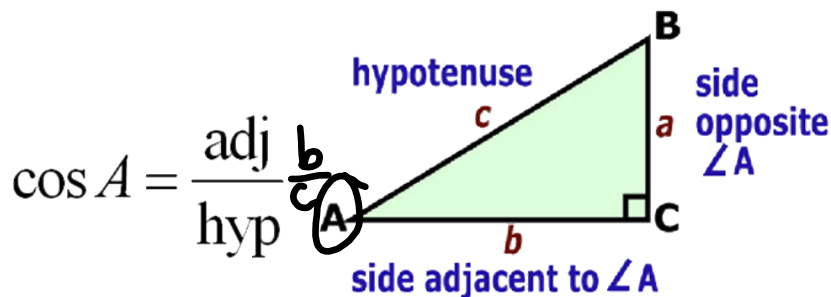
If we have a right triangle with acute angle A:



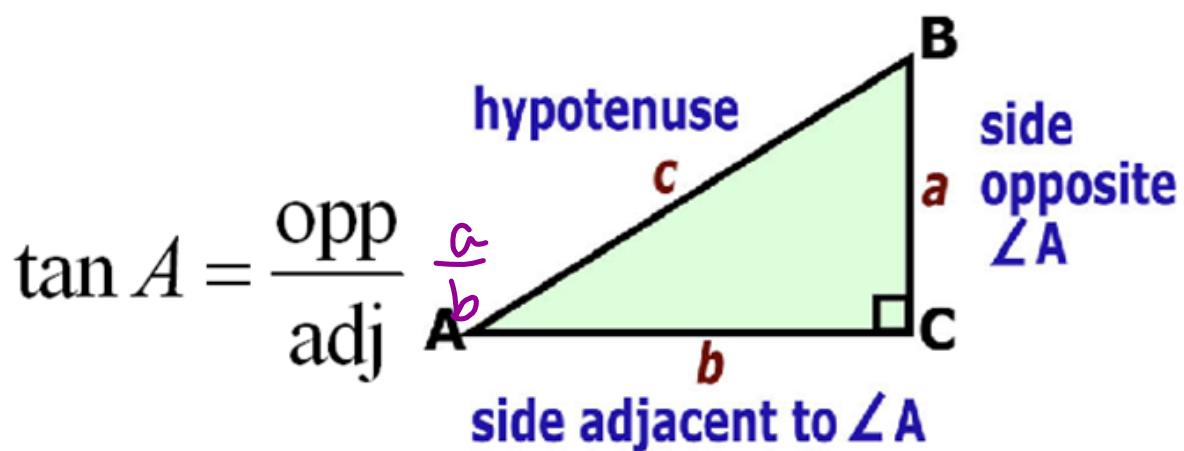
sine- the ratio between the length of opposite side to that of the hypotenuse



cosine- The ratio between the length of adjacent side to that of the hypotenuse.



tangent - The ratio between the length of opposite side to that of the adjacent side.



SOH CAH TOA

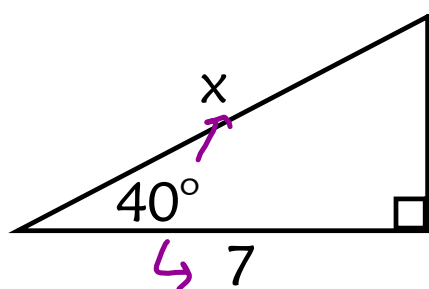
When using trigonometry, it is  
VERY IMPORTANT

that your calculator be in  
DEGREE MODE! ✖

(You will do radians a little later on)

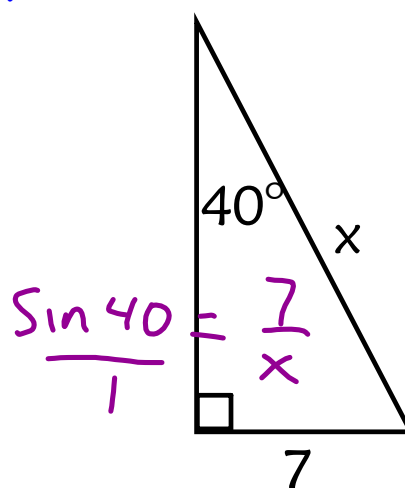
Go to MODE,  
then down to where it  
says radian/degree  
and  
highlight degree.

Find the missing side length.  
(assume right triangles)



$$\frac{\cos 40}{1} = \frac{7}{x}$$

$$7 = \frac{x \cos 40}{\cos 40}$$



$$\frac{\sin 40}{1} = \frac{7}{x}$$



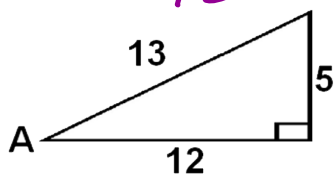
## Inverse trig functions:

$\sin^{-1}$ ,  $\cos^{-1}$ ,  $\tan^{-1}$

- Inverses are used to find the degree measure of an angle.  
(Sin, Cos and Tan find the side lengths.)
- An inverse function will negate ("undo") a trig function.

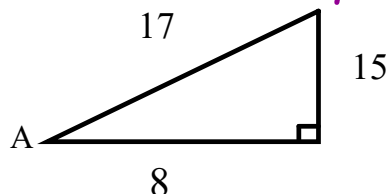
What is the measure of angle A?

~~$\tan^{-1}$~~   $\tan^{-1}$   
 ~~$\tan A = \frac{5}{12}$~~



$A = \tan^{-1} \frac{5}{12}$

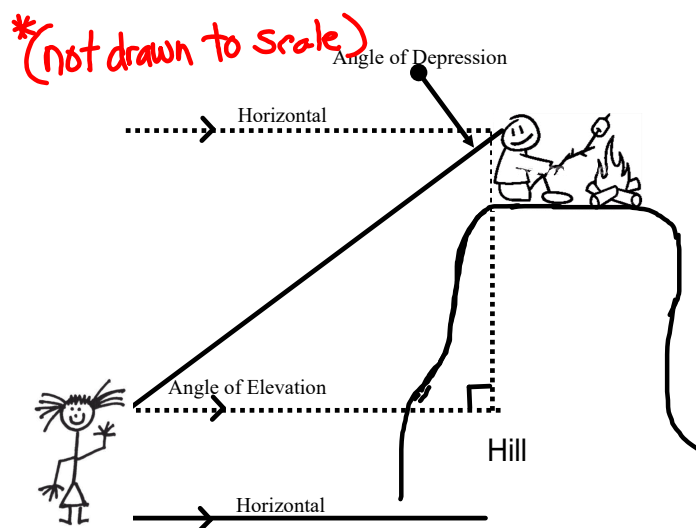
~~$\cos^{-1}$~~   $\cos^{-1}$   
 ~~$\cos A = \frac{8}{17}$~~



Our situation:

A girl standing at the bottom of a hill is waving up to her friend who is roasting marshmallows at a campfire on the top of the hill.

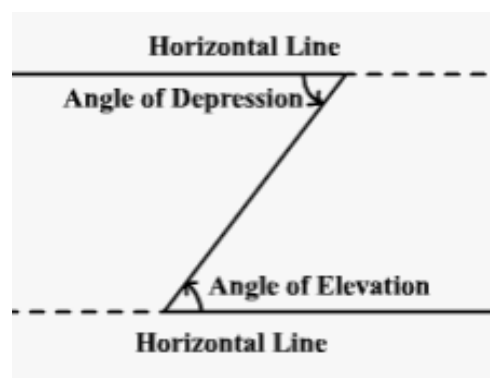
If the angle of elevation is  $27^\circ$  and the girl is 200 feet from the bottom of the hill, approximately how tall is the hill?



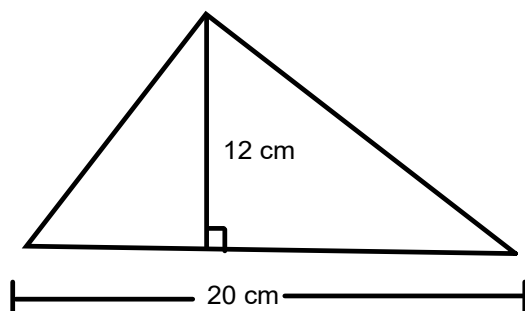
Angle of Elevation: If you are looking up, it is the angle from the horizontal UP to the line of sight.

Angle of Depression: If you are looking down, it is the angle from the horizontal DOWN to the line of sight.

Why does this work?



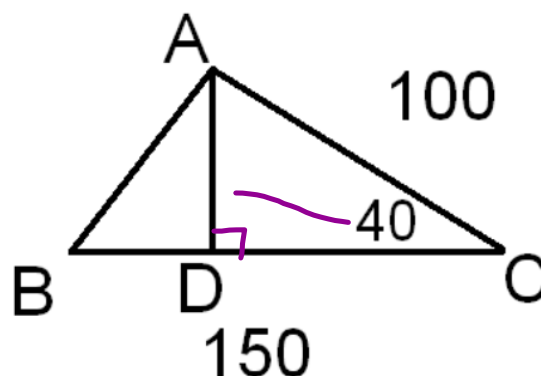
How can we find the area of the triangle?



$$\frac{20 \cdot 12}{2}$$

$$\sin 40 = \frac{AD}{100}$$

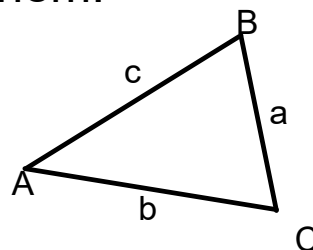
How about this one?



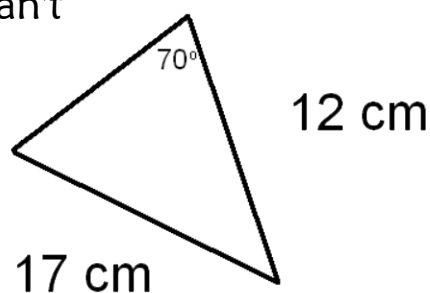
The area of an acute triangle is given by  
the formula

$$A = \frac{1}{2} ab \sin C$$

where  $a$  and  $b$  are the lengths of two sides and  $C$  is the acute angle between them.

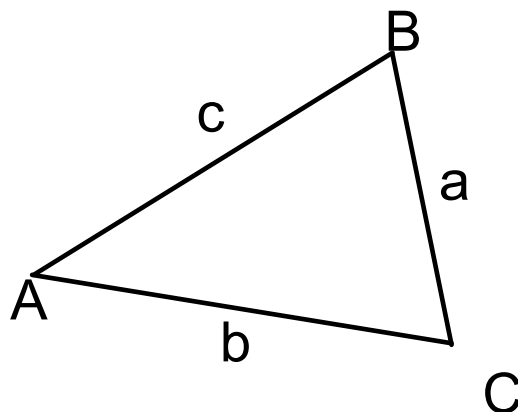


How can we solve for the missing side and angles here? Please note that we can't use Sin, Cos or Tan as this isn't marked as a right triangle (and we can't make that assumption!)

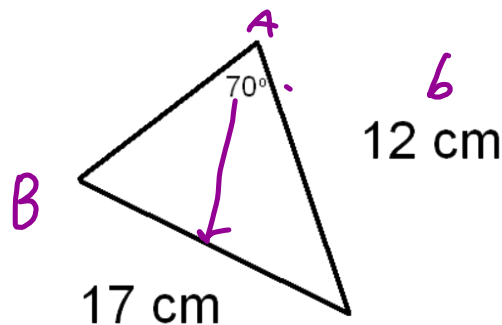


Law of Sines -For an acute triangle with angles of measures A, B and C and sides of lengths a, b and c (a opposite from A, b opposite from B, and c opposite from C),

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Let's solve for the missing side and angles.

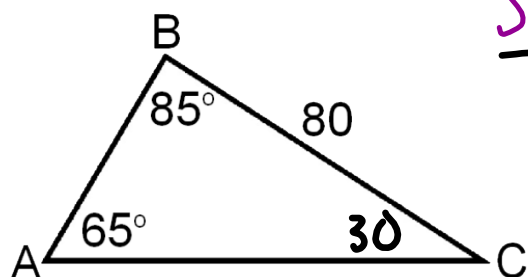


$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 70}{17} = \frac{\sin B}{12}$$

$$\sin^{-1} \left( \frac{12 \sin 70}{17} \right) = \frac{12}{17} \sin B$$

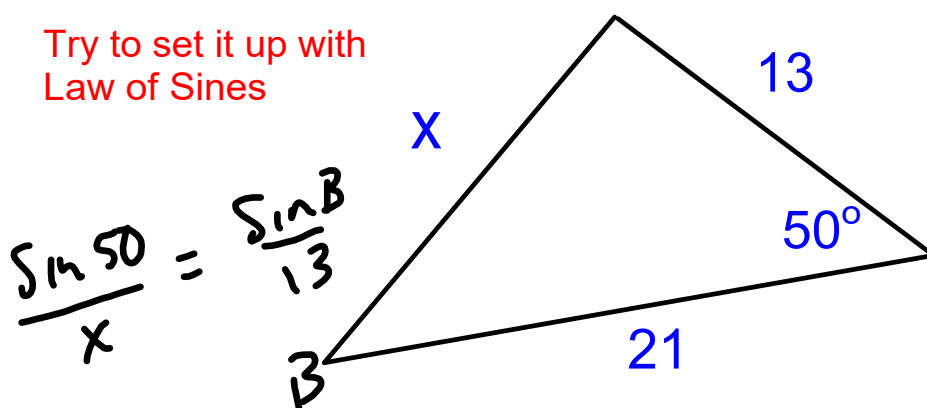
Let's solve for the missing side and angles.



$$\frac{\sin 85}{AC} = \frac{\sin 65}{80}$$

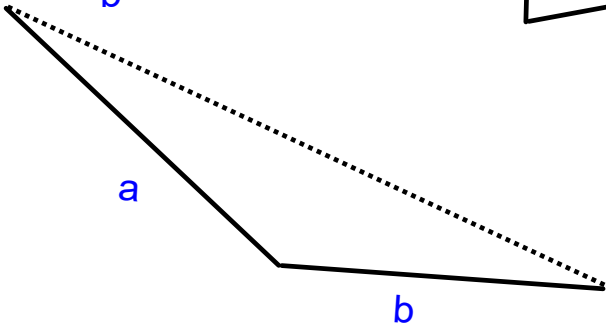
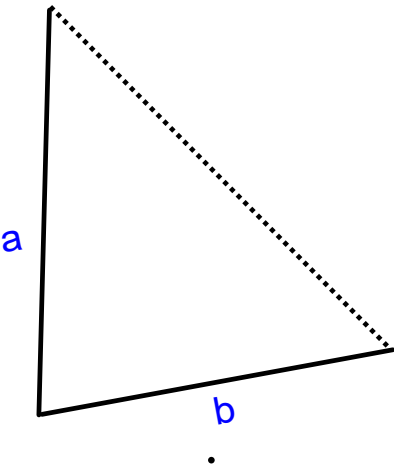
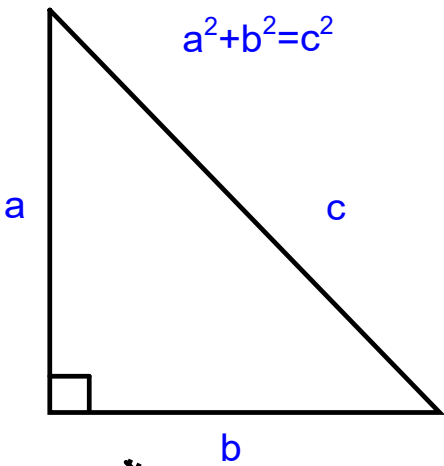
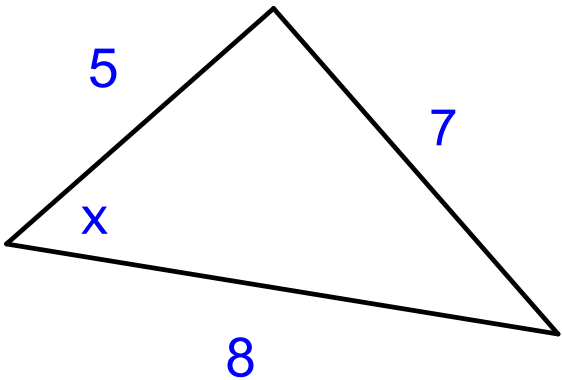
Try to solve a situation like this...

Try to set it up with  
Law of Sines



$$\frac{\sin 50}{X} = \frac{\sin B}{13}$$

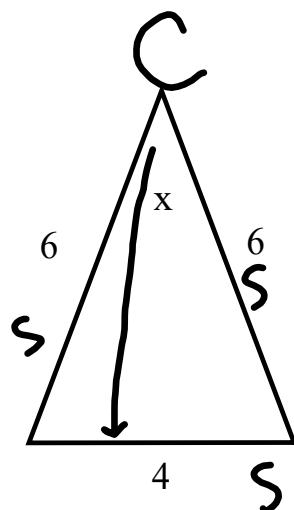
Or This...



In order to solve those triangles you need  
the... **Law of Cosines!**

$$c^2 = a^2 + b^2 - 2ab \cos C$$

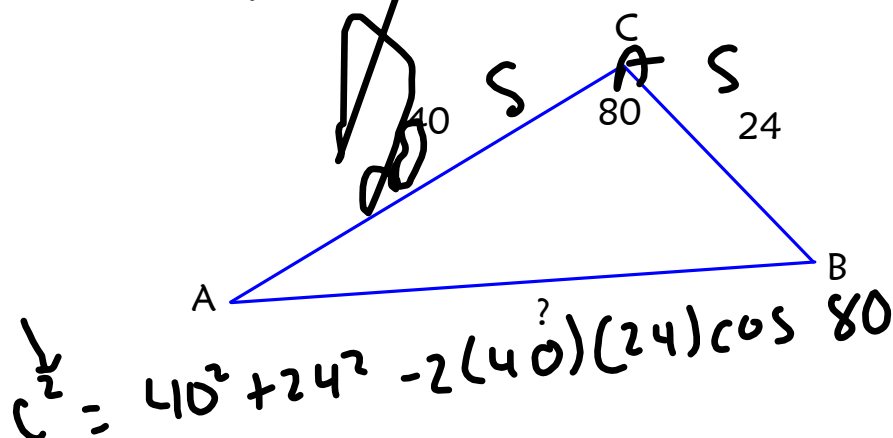
Use law of Cosines to find the missing angle.



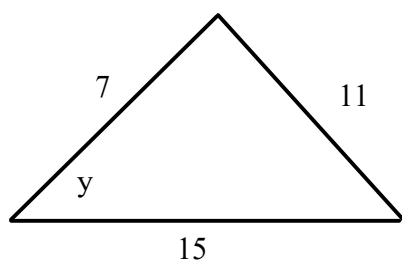
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ 4^2 &= 6^2 + 6^2 - 2(6)(6) \cos C \\ 16 &= 72 - 72 \cos C \\ -56 &= -72 \cos C \\ \frac{-56}{-72} &= \frac{-72 \cos C}{-72} \end{aligned}$$



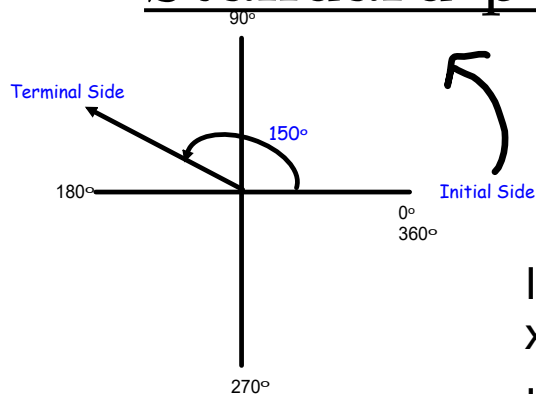
Use law of Cosines to find the missing side.



Use law of Cosines to find the missing angle.



## Standard position of an angle



Its **initial side** is along the positive x-axis.

It is measured by rotating from the **initial side** to the **terminal side** in a **counterclockwise** direction.

## Examples of angles in standard position

Positive

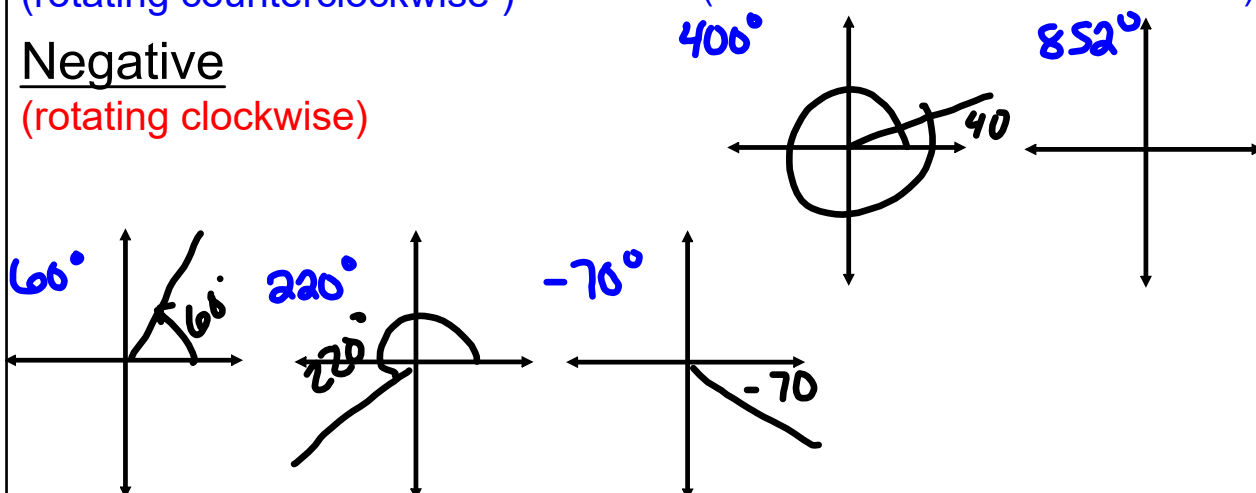
(rotating counterclockwise )

Negative

(rotating clockwise)

Angles greater than  $360^\circ$

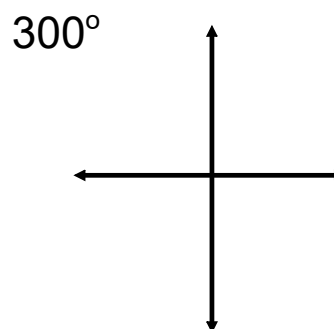
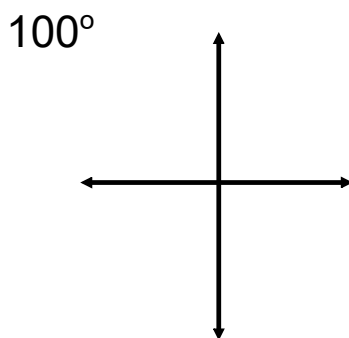
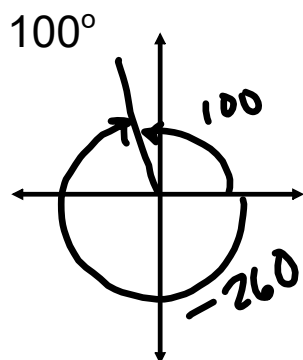
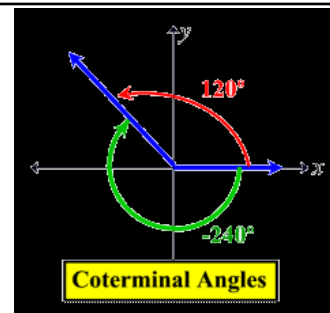
(revolve around more than once!)



## Coterminal Angles

Angles that share the same terminal side.

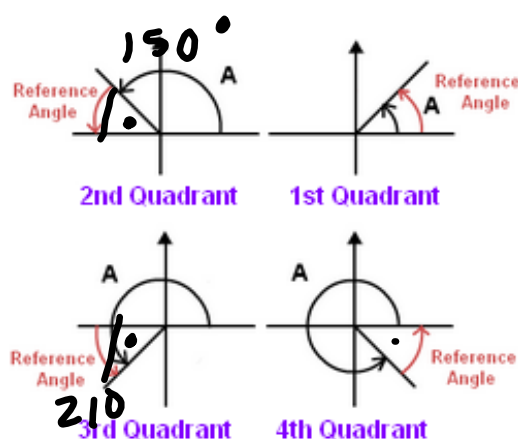
ex) Draw and name the coterminal angle to the given angle.



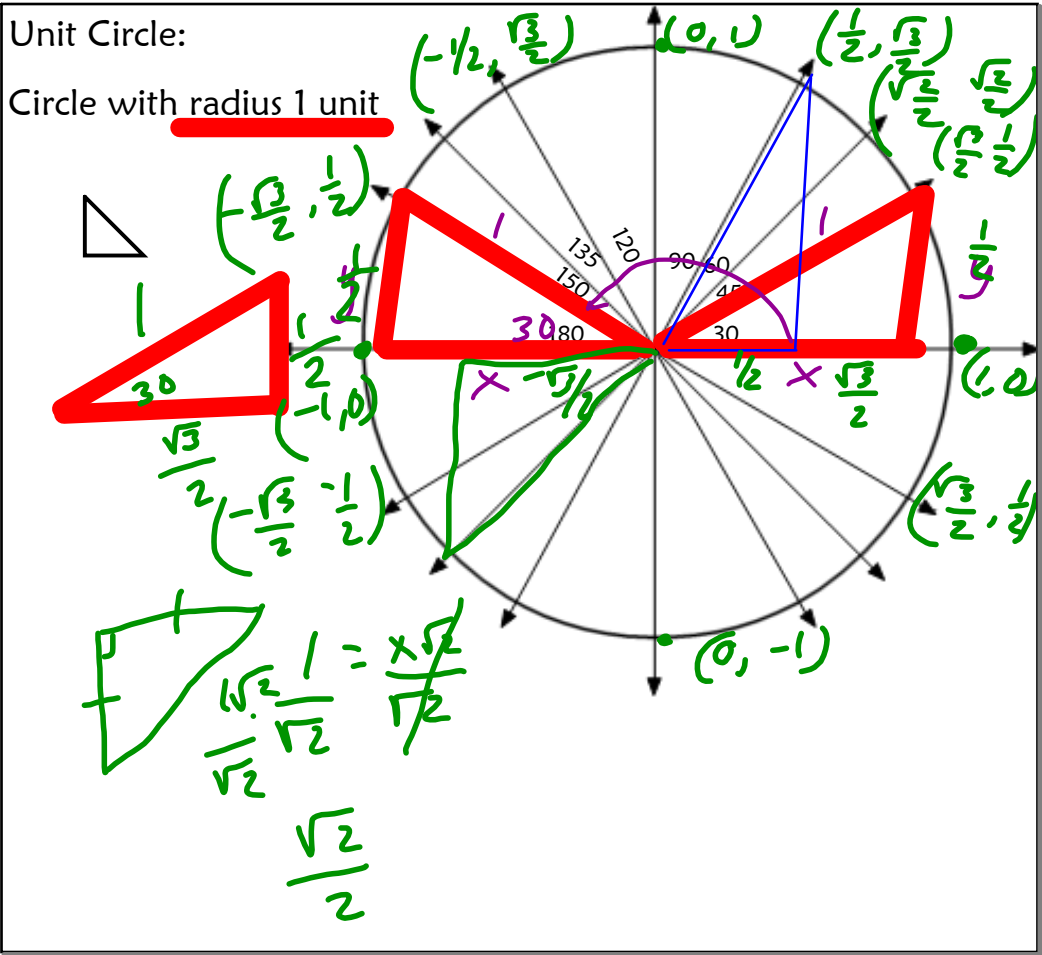
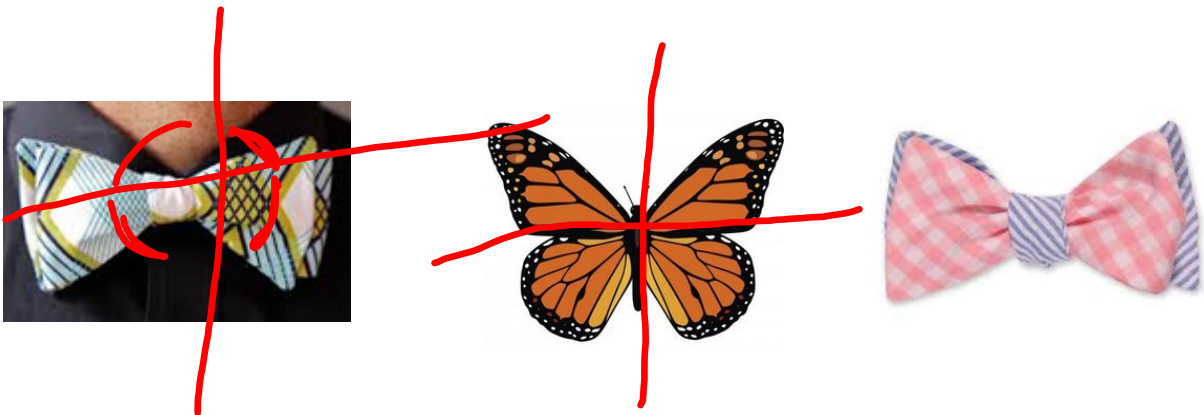
## Reference Angle

The positive, acute angle between the horizontal axis and the terminal side.

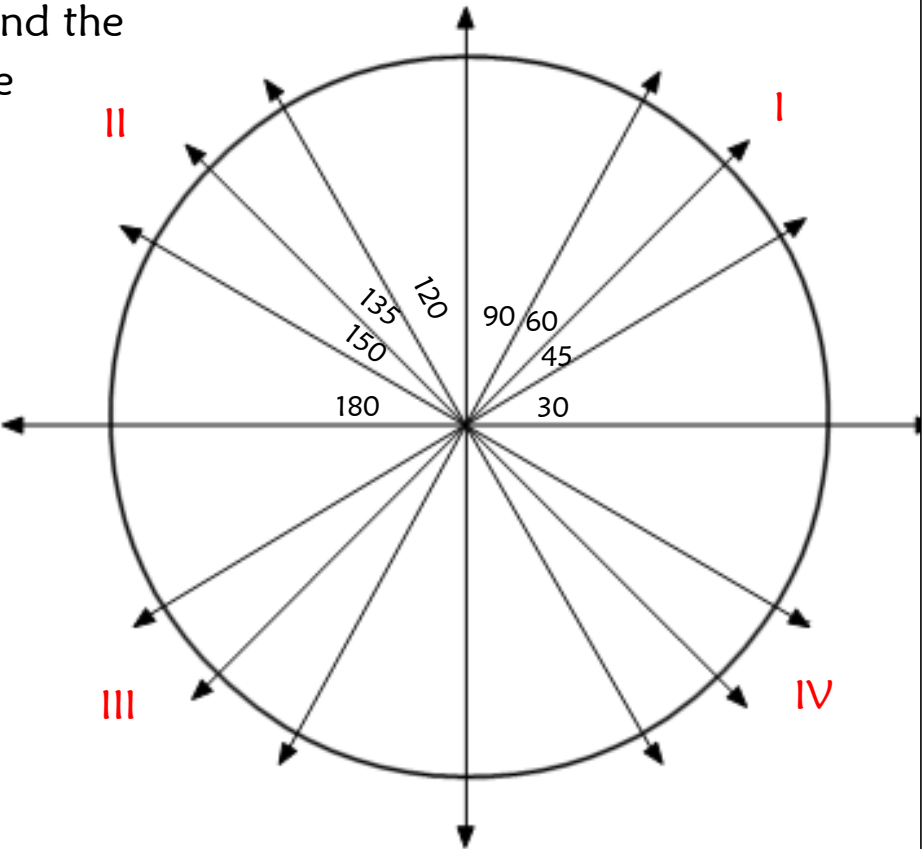
AKA the angle it makes with the x-axis.



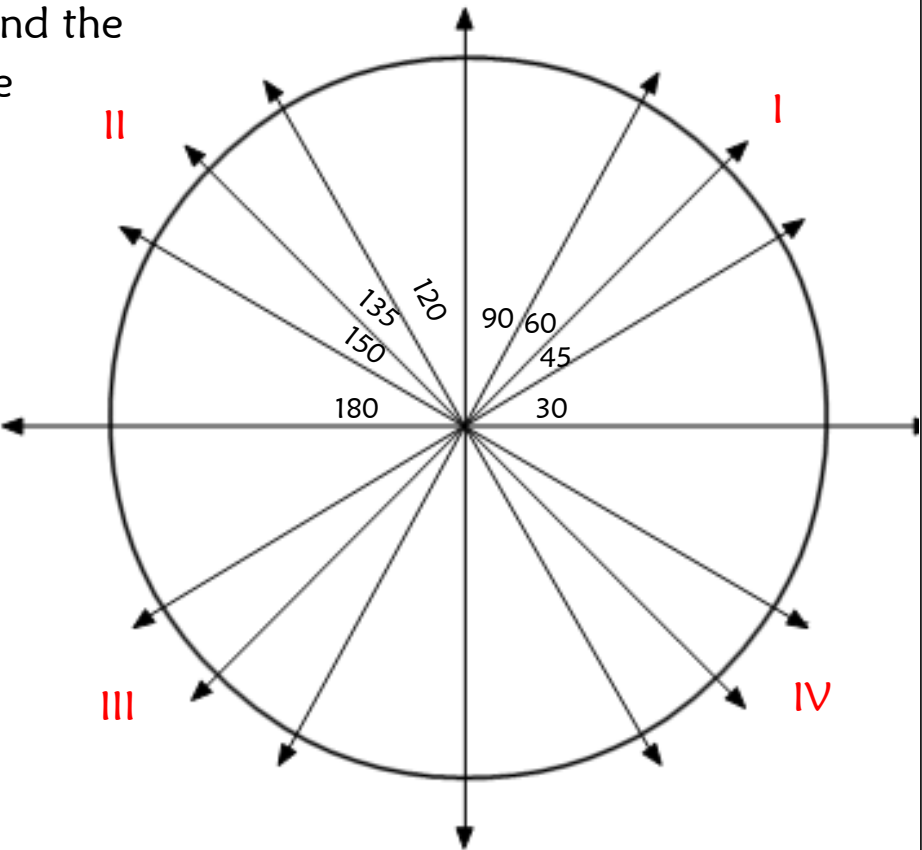
think butterflies and bow ties



In your groups, find the side lengths of the triangles in your quadrant.



In your groups, find the side lengths of the triangles in your quadrant.

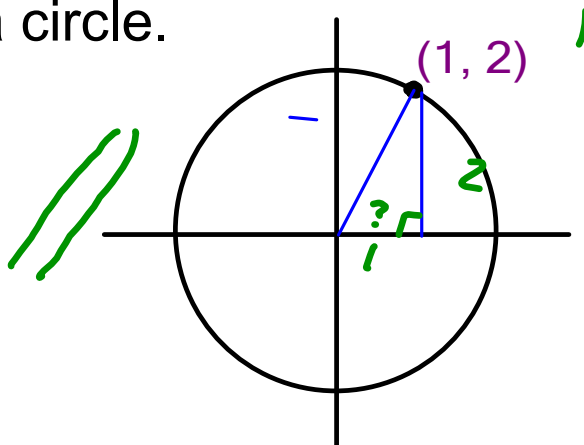


## Defining the Circular Function

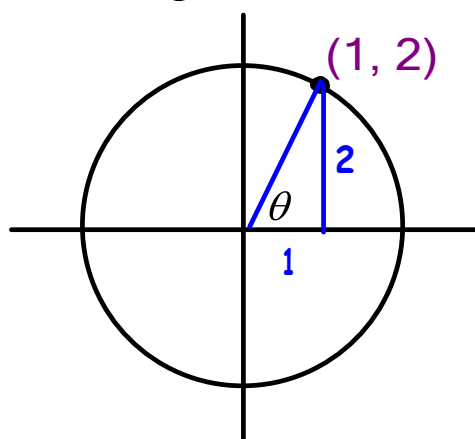
Making the link between the circle and trig.

Goal: Find the reference angle if the point (1, 2) is on the edge of a circle.

$$\tan \theta = \frac{2}{1}$$



Find the reference angle if the point (1, 2) is on the edge of a circle.



SOH CAH TOA

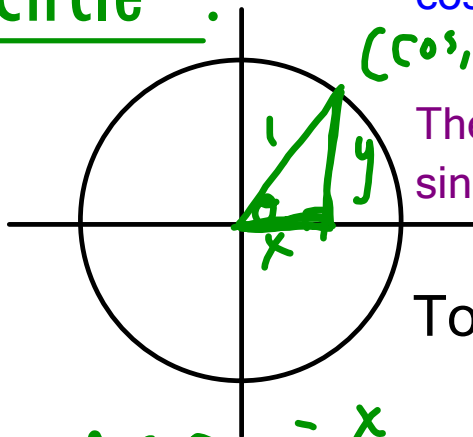
$$\tan \theta = \frac{2}{1}$$

$$\theta = \tan^{-1} \left( \frac{2}{1} \right)$$

$$\theta \approx 63.4^\circ$$

# The Unit

## Circle :



The x-coordinate of any point on the unit circle is simply the cosine of the angle!

The y-coordinate is simply the sine of the angle!

To find x:

To find y:

$$\cos \theta = \frac{x}{1}$$

$$\sin \theta = \frac{y}{1}$$

Remember:

x y  
(cos θ, sin θ)

★

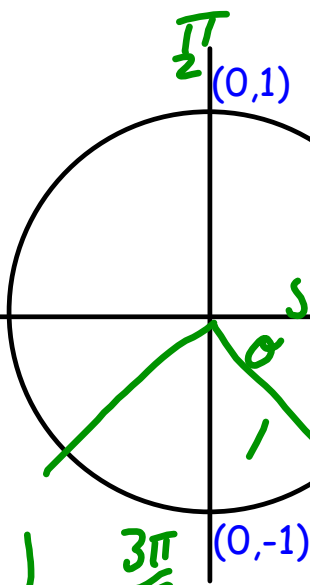
$$\cos 180^\circ =$$

$$\sin 180^\circ =$$

$$\frac{\cos 225^\circ}{\sin 225^\circ} = 1$$

$$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

π radians



$$\cos 90^\circ = 0$$

$$\sin 90^\circ = 1$$

$$\cos 0^\circ = 1$$

$$\sin 0^\circ = 0$$

$$\cos 360^\circ =$$

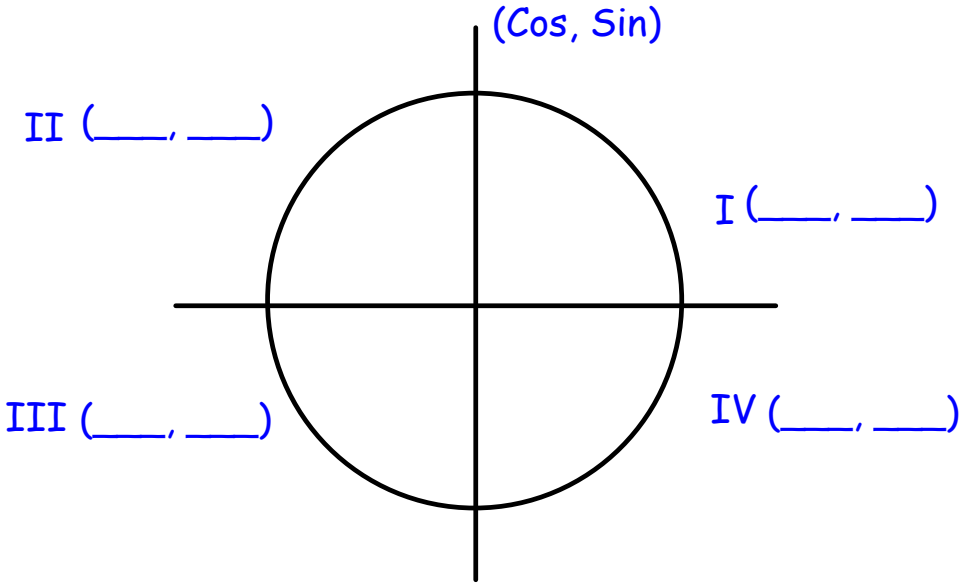
$$\sin 360^\circ =$$

$$\cos 270^\circ =$$

$$\sin 270^\circ =$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}} = \frac{\text{hyp}}{\text{hyp}}$$

# Where are Cosine and Sine Positive/Negative??



## Practice with Law of Sines & Law of Cosines

- 1) A ranger in fire tower A spots a fire at a direction of  $40^\circ$ . A ranger in fire tower B, which is 28 miles directly east of tower A, spots the same fire at a direction of  $116^\circ$ . How far from tower A is the fire?
- 2) A guy wire to a tower makes a  $65^\circ$  angle with level ground. At a point 39 ft farther from the tower than the wire but on the same side as the base of the wire, the angle of elevation to the top of the tower is  $36^\circ$ . Find the length of the wire (to the nearest foot).
- 3) A boat leaves the dock and sails in a direction of  $70^\circ$ . Once reaching its destination on the opposite shore, it sails in a direction of  $272^\circ$  and docks 150 km north of its original starting position. What is the total distance the boat has traveled?
- 4) Two factories blow their whistles at exactly the same time. If a man hears the two blasts exactly 2.3 seconds and 6.7 seconds after they are blown and the angle between his lines of sight to the two factories is  $46.4^\circ$ , how far apart are the factories? Give your result to the nearest meter. (Use the fact that sound travels at 344 m/sec.)
- 5) A parallelogram has sides of length 36.4 cm and 21.5 cm. If the longer diagonal has a length 38.9 cm of what is the angle opposite this diagonal? Give your answer to the nearest tenth of a degree.
- 6) A ship travels 98 km on a bearing of  $38^\circ$ , and then travels on a bearing of  $128^\circ$  for 169 km. Find the distance of the end of the trip from the starting point, to the nearest kilometer.