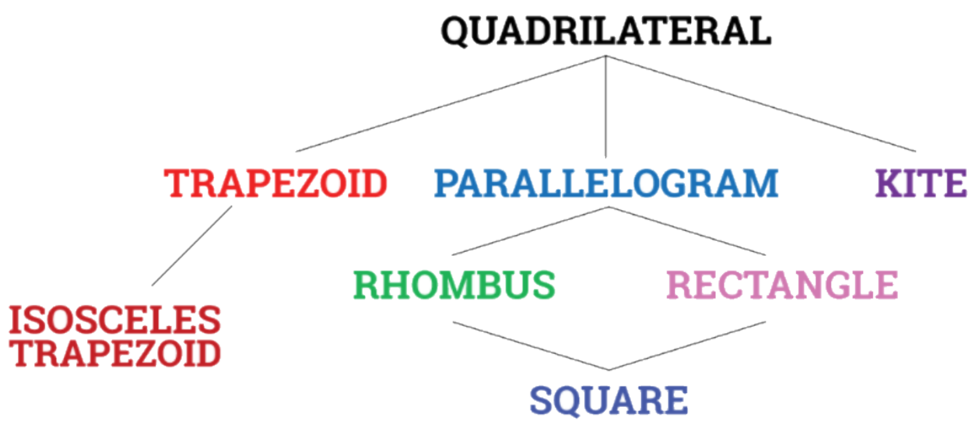
UMTYMP Geometry Day 8

Chapter 9 Polygons

~It's so good to see you!! I'm glad to be back!

Quadrilaterals

Quick reminder from last week...



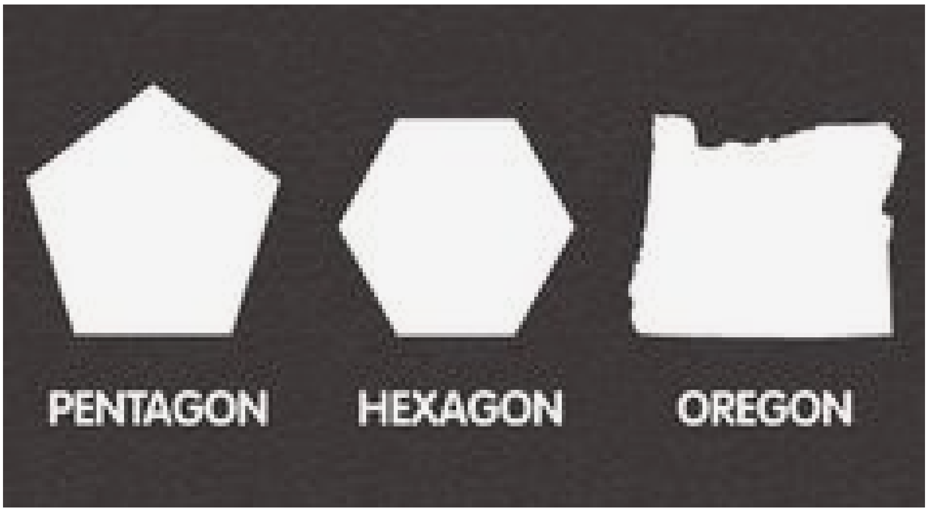
Polygons

You probably know some of this unit already...

- Number of sides/names
- Interior/exterior angle sums
- Polygon area

# of sides	Polygon Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon/Septagon
8	Octagon
9	Nonagon
10	Decagon
12	Dodecagon
n	

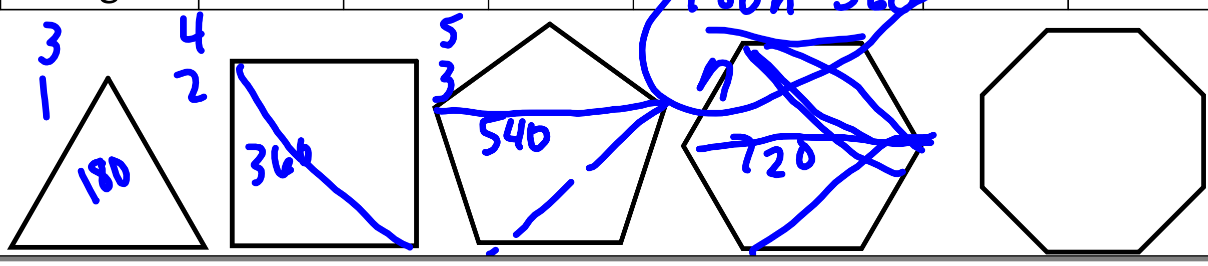
Or in Minnesota...



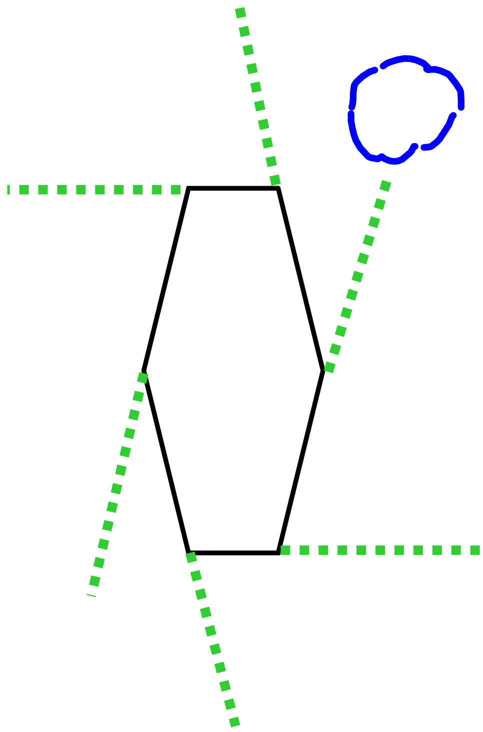
Regular polygons are NOT boring...they are very special!

Interior angle sum:

Sides	3	4	5	6	8	...	n
Interior Angle Sum	180	360	540	720	1080		180(n-2)
EACH Interior Angle	60	90	105	120	135 $\frac{180n-360}{n}$		$\frac{180(n-2)}{n}$



Exterior angles...



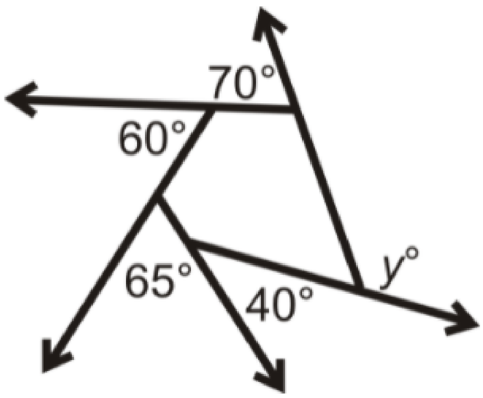
$$a+b+c=180$$
$$-a \qquad\qquad -a$$
$$b+c=180-a$$

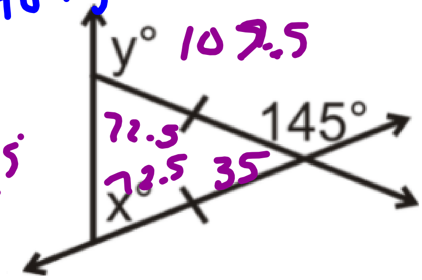
The sum of the exterior angles of any polygon is... 360°

Exterior angles...

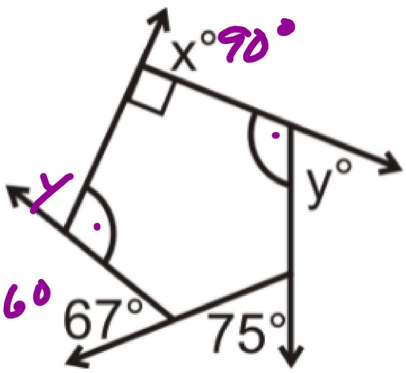
What is y?

$70+60+65+40+y=360$



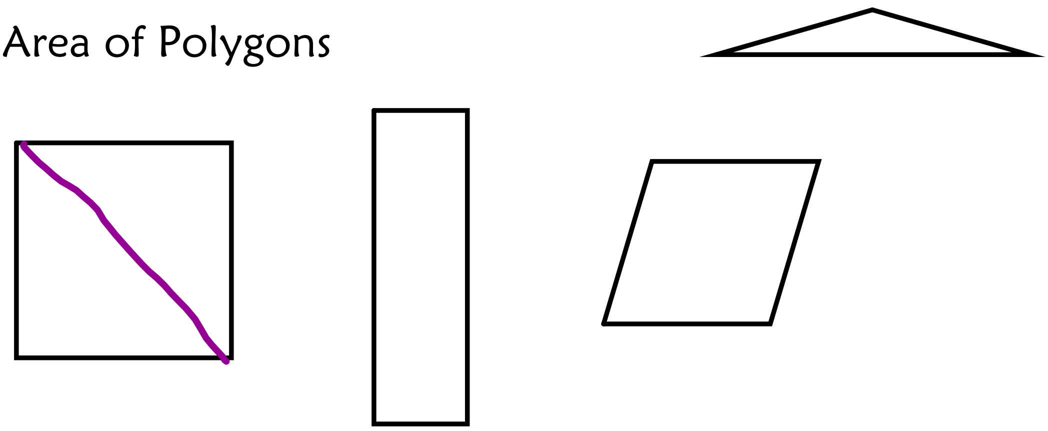


$\frac{145}{2}$

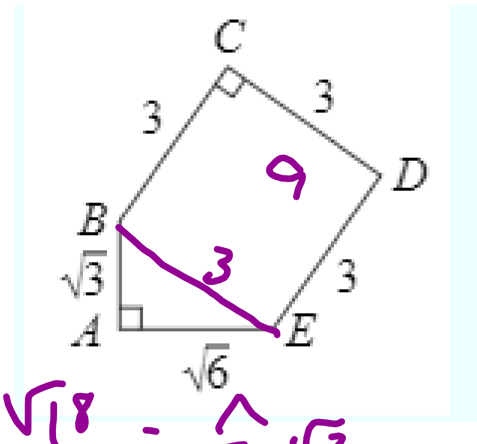


$90+2y+67+75=360$

Area of Polygons



Find the area of pentagon ABCDE given the information shown.



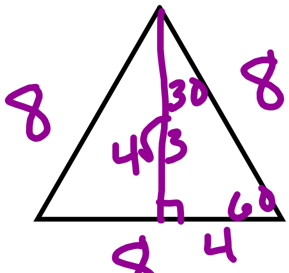
$\frac{3\sqrt{2}}{2}$

$9 + \frac{3\sqrt{2}}{2}$

$9 + \frac{\sqrt{3}\sqrt{6}}{2}$

$\frac{\sqrt{18}}{2} = \frac{\sqrt{9}\sqrt{2}}{2}$

Find the area of a regular hexagon with side length 8.

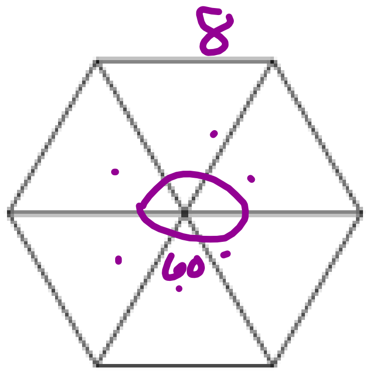


$\frac{8 \cdot 4\sqrt{3}}{2}$

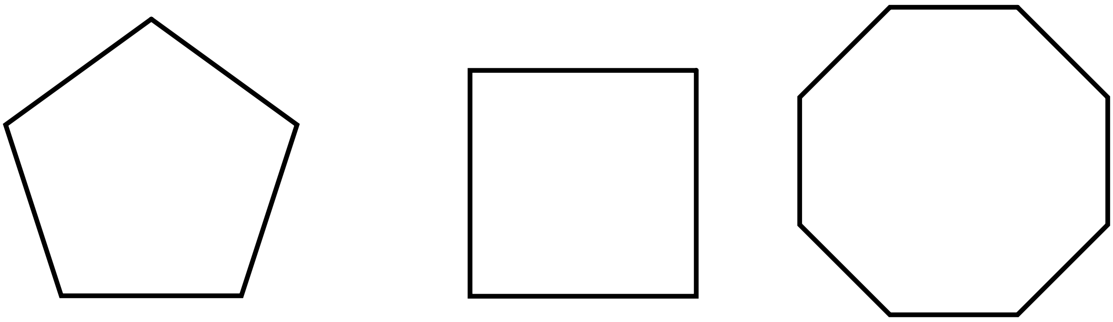
$(16\sqrt{3})\ 6$

$96\sqrt{3}$

$\frac{1}{2}\ 4\sqrt{3} \cdot 8 \cdot 6$



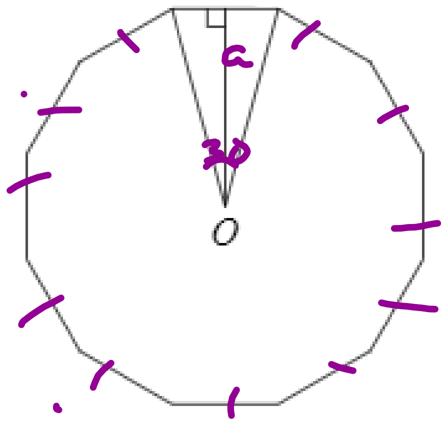
Area of Regular Polygons



More Area of Regular Polygons:

A=1/2 asn or A = 1/2aP

$\frac{1}{2}P \cdot a$



where a = apothem

s = side length

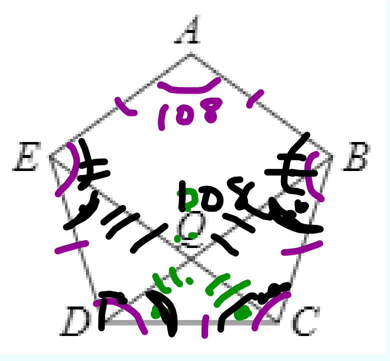
n = number of sides

P = perimeter

Important: The area of a regular polygon is half its perimeter times the distance from the center of the polygon to a side.

Problem 9.10

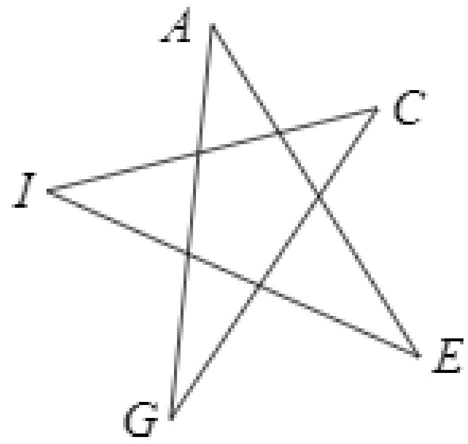
Diagonals $\overline{BD}$ and $\overline{CE}$ of regular pentagon $ABCDE$ meet at Q. Find $\angle BQE$.



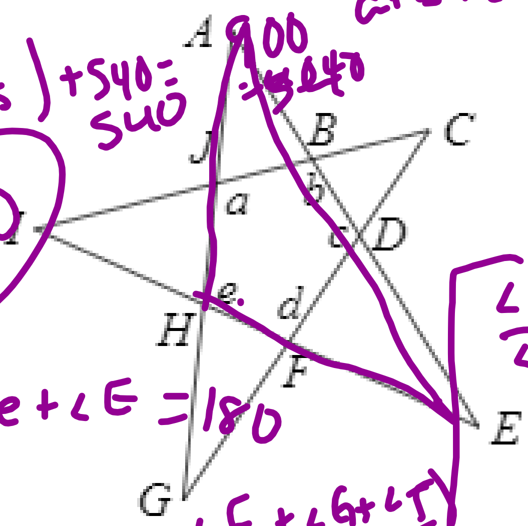
In the star diagram, find the sum

$$<A + C + <E + <G + <I$$

(You cannot assume the pentagon in the middle of the star is regular!)



Stuck? Maybe this will help...



$a+b+c+d+e=540$

$180 \cdot 5 - 720$

$(180)5 - 360$

$2(\angle s) + 540 = 900$
$-540 \quad -540$

360

180

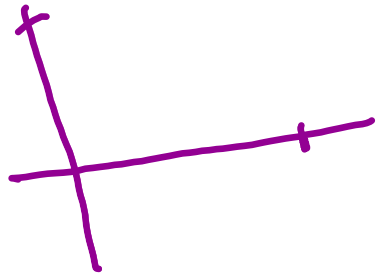
$\angle C + a + \angle G = 180$
$\angle E + b + \angle I = 180$
$\angle G + c + \angle A = 180$
$\angle I + d + \angle C = 180$
$\angle A + e + \angle E = 180$

$\angle A + e + \angle E = 180$

$2(\angle A + \angle C + \angle E + \angle G + \angle I) + a + b + c + d + e = 5(180)$

Constructions: Regular Polygons

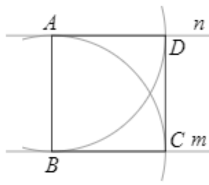
Construct a square



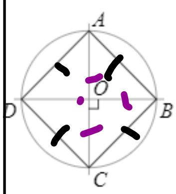
Construct a regular hexagon

Construct a square Problem 9.15 (page 258)

Solution for Problem 9.15: Solution 1: A natural solution is to start with one side, $\overline{AB}$, then construct the others. The other two vertices must be on lines through A and B that ~~are~~ perpendicular to $\overline{AB}$, so we construct lines m and n perpendicular to $\overline{AB}$ as shown. We can then find C on m such that $AB = BC$ by constructing a circle with center B and radius AB. Where this circle meets m gives us C. Similarly, we construct a circle with center A and radius AB to find D on n such that $AD = AB$.

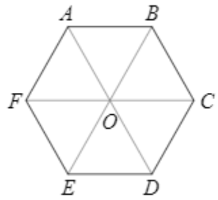


We'll leave the proof that this construction does indeed result in $ABCD$ being a square for an Exercise.



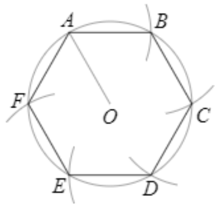
Solution 2: We can find an even slicker construction by thinking a little more about squares. For example, we know that the diagonals of a square are perpendicular and they meet at a point that is equidistant from all four vertices. In other words, the intersection of the diagonals of a square is the center of the circumcircle of the square. We can use this observation to start with a circle to make our square. We draw a circle with center O. We then draw two perpendicular lines through O and label the points where these lines hit the circle A, B, C, and D. Since the four triangles that meet at O are congruent 45-45-90 triangles, we have $AB = BC = CD = DA$. Moreover, each angle of $ABCD$ equals $45^\circ + 45^\circ = 90^\circ$. Therefore, $ABCD$ is a square. □

Construct a regular hexagon Problem 9.16 (page 258)



Solution for Problem 9.16: Solution 1: We already know how to make an equilateral triangle, and a regular hexagon can be built from six equilateral triangles. Therefore, we can start with equilateral $\triangle AOB$, then construct equilateral $\triangle OBC$ on $\overline{OB}$, then construct equilateral $\triangle OCD$ on $\overline{OC}$, and so on. After making 6 equilateral triangles, we'll have regular hexagon $ABCDEF$.

But that's an awful lot of work. There must be a faster way.



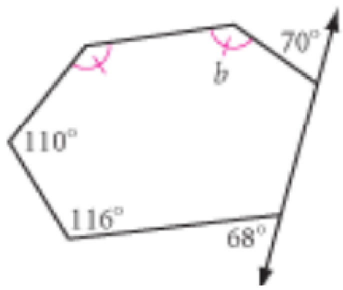
Solution 2: Since the circumcircle worked so well with the square, we'll try it with the hexagon. Since the long diagonal of a regular hexagon is equal both to twice the side length of the hexagon and to the diameter of the circumcircle of the hexagon, the radius of the circumcircle equals the length of a side of the regular hexagon. Therefore, if we draw a circle with radius $\overline{OA}$, the vertices of a regular hexagon inscribed in the circle will be at intervals of length OA around the circle.

So, we draw an arc with center A and radius OA to find point B on the circle. Then we draw an arc with center B and radius OA to get C, and so on around the circle. The resulting $ABCDEF$ is a regular hexagon.

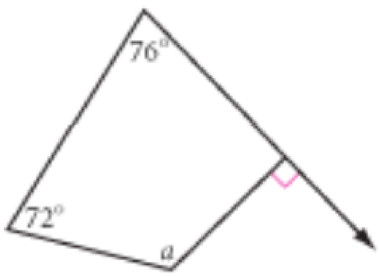
We can prove $ABCDEF$ is regular by noting that each of the triangles formed by connecting consecutive vertices to the center is equilateral. For example, $AO = BO = AB$ means $\triangle AOB$ is equilateral. Similarly, so is $\triangle BOC$, so $\angle ABC = 2(60^\circ) = 120^\circ$. In the same way, we see that each of the angles $ABCDEF$ is 120°, and each of the sides is equal (to the radius of the circle). Therefore, $ABCDEF$ is indeed regular. □

Practice Problems:

$b = \underline{?}$

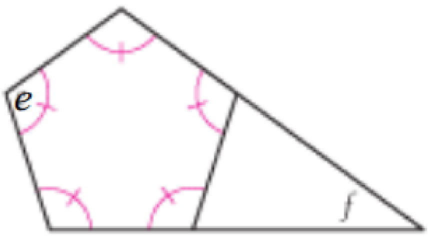


$a = \underline{?}$



$e = \underline{?}$

$f = \underline{?}$



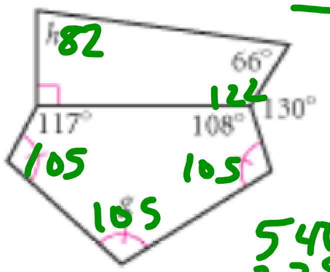
Practice Problems:

8

$g = \underline{?}$ (h)

$h = \underline{?}$

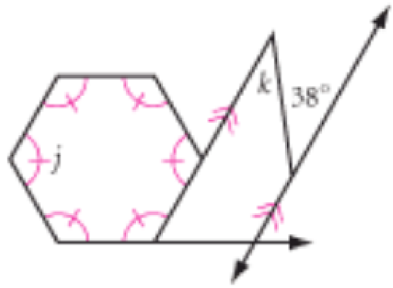
$$\begin{array}{r} 360 \\ -238 \\ \hline \end{array}$$

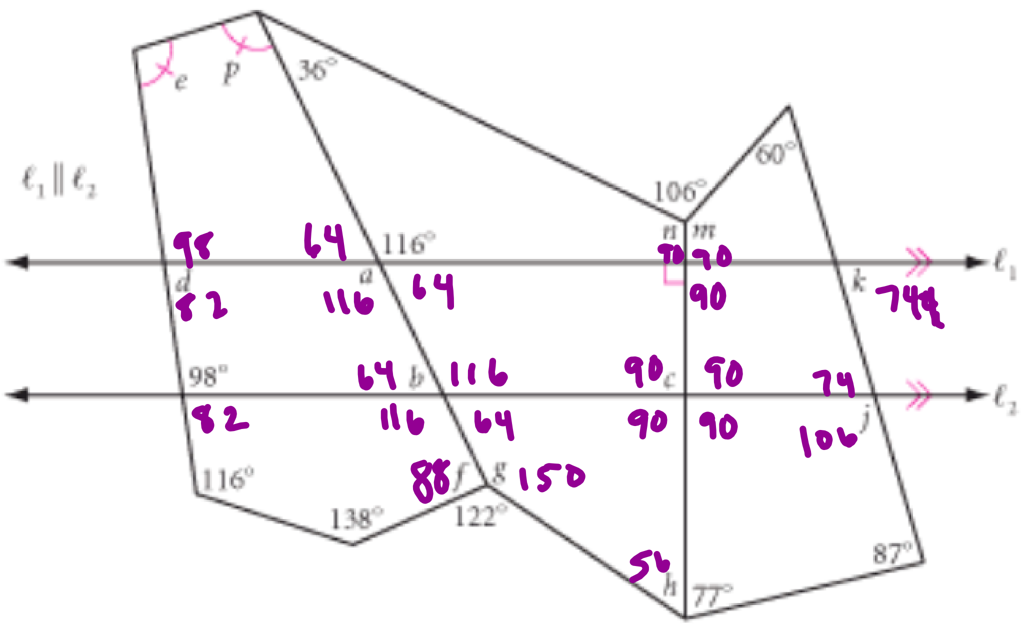


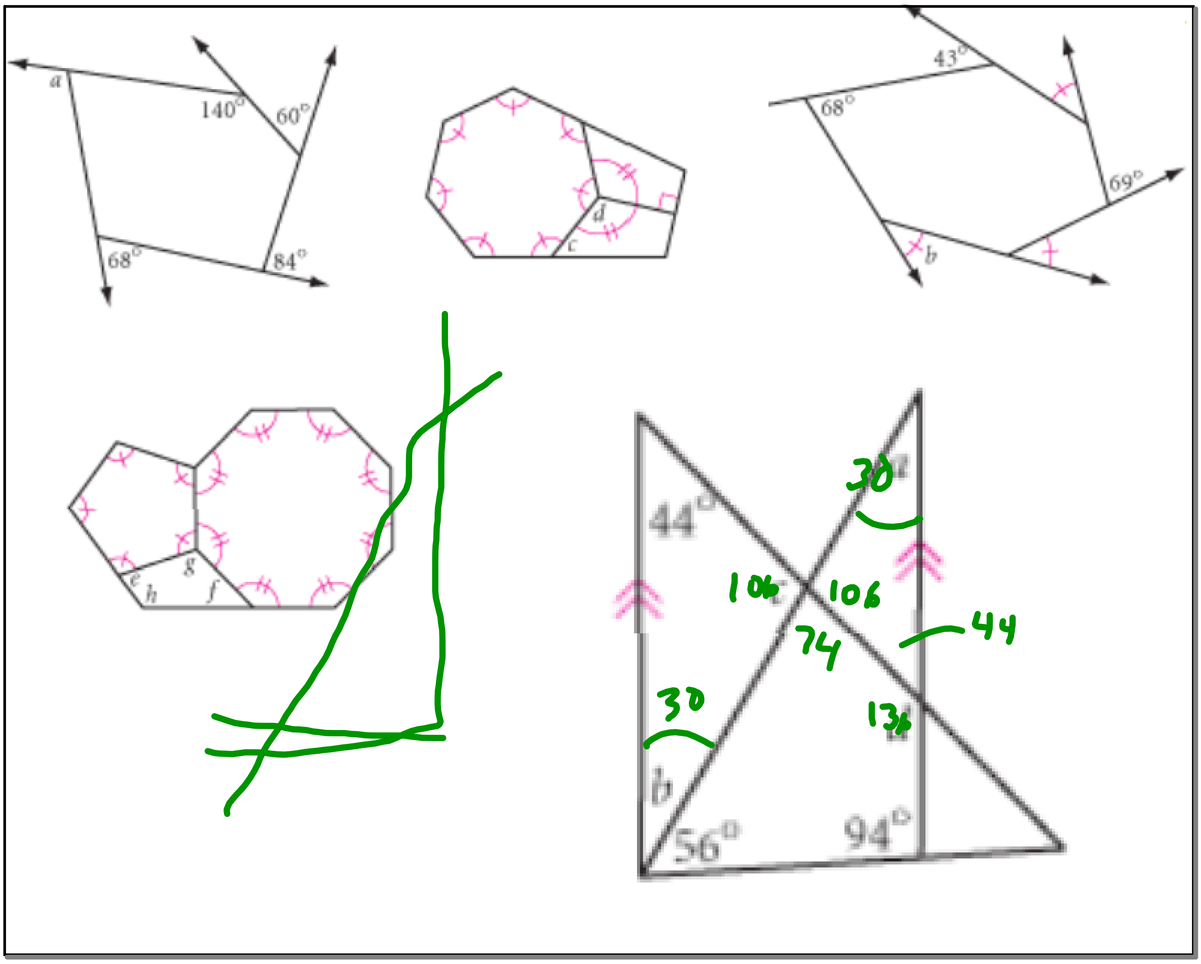
$$\begin{array}{r} 540 \\ -225 \\ \hline 315 \end{array}$$

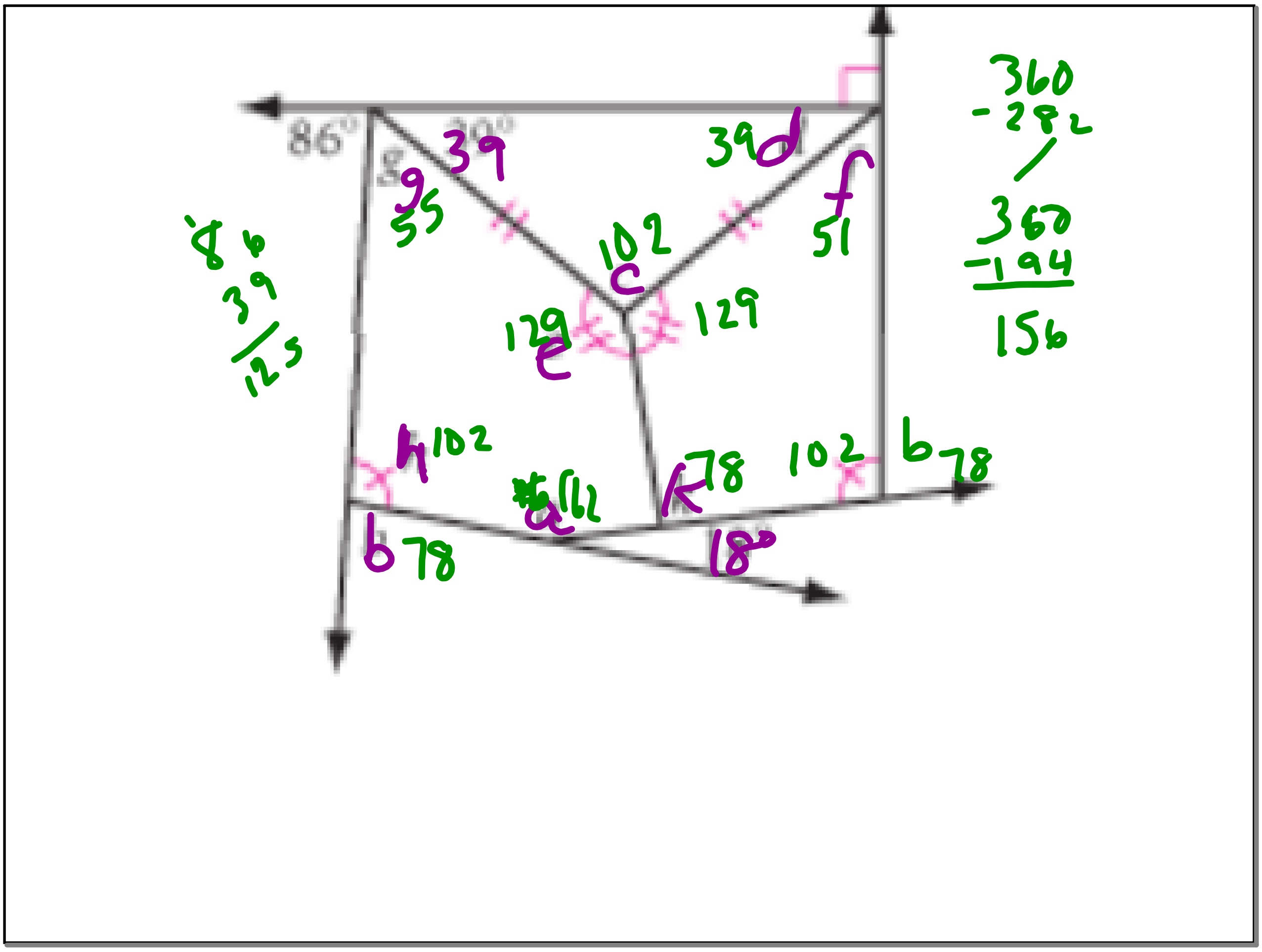
$j = \underline{?}$

$k = \underline{?}$






a
140°
60°
68°
84°
d
c
43°
68°
69°
b
g
e
h
f
44°
56°
94°
b
30
106
106
74
44
30
136


86°
39°
g 39
55
39 d
f
51
102
c
129
129
e
h 102
a 162
k 78
102
b 78
b 78
18°
86
39
125
360
- 282
360
-194
156

Summary:

Definitions: A **polygon** is a closed planar figure with line segments as boundaries. As with triangles and quadrilaterals, the segments that form the boundaries are the **sides**, which meet at the **vertices** of the polygon. A **diagonal** is a segment that is not a side, but connects two vertices of a polygon. A **regular polygon** is a polygon in which all the angles have the same measure and all the sides have the same length.

Important:

- The area of a regular polygon is half its perimeter times the distance from the center of the polygon to a side.
- In a polygon with n sides, the measures of the interior angles have a sum of $(n-2)(180^\circ)$ and the measures of the exterior angles have a sum of 360°.

Problem solving strategies:

Problem Solving Strategies

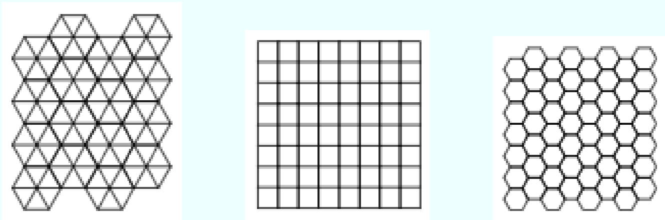
Concepts:

- Sometimes thinking about the exterior angles of a polygon offers a simpler approach than thinking about the interior angles!
- Break complicated areas into pieces you can handle. Sometimes we can view a desired area as the 'leftover' portion from having simple pieces taken away from a simple starting figure.
- Many problems involving regular hexagons can be tackled by dissecting the hexagons into equilateral triangles.
- Symmetry can be a very useful tool in problems involving regular polygons.
- Most polygon problems are essentially quadrilateral and triangle problems. When stuck with a polygon, try dissecting it into quadrilaterals and triangles.
- The circumcircle of a regular polygon can be extremely useful in working with the polygon.

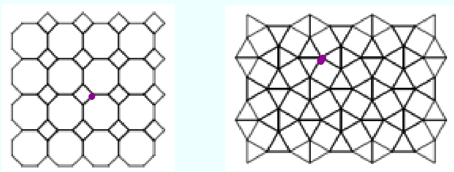
Have you ever seen a bathroom floor tiled with regular pentagons? How about with regular hexagons? Heptagons? Why do squares, triangles, and hexagons seem to show up everywhere, while the other regular polygons don't get much use?

Like lots of other things in geometry, it all boils down to angles. Now that you know how to find the angle in a regular polygon, you can answer this problem for yourself! Take a look at a tiling that does work, such as covering the plane with squares (like a piece of graph paper). At each vertex, exactly four squares meet. The angle of each square is 90°, so the four angles at each vertex add up to 360°.

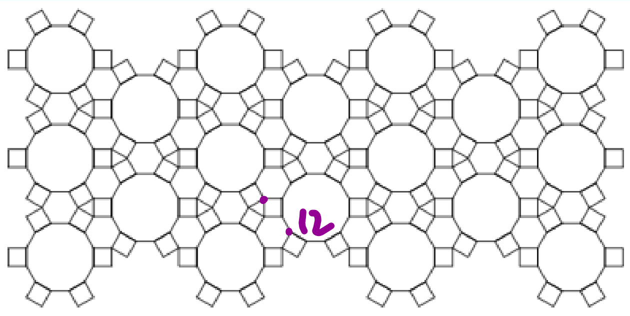
If you solved Problem 9.46, you should be able to quickly identify which regular polygons can be used to tile a plane all by themselves. We call such a regular **tiling** of the plane a **tessellation**. The three tessellations that use only a single regular polygon over and over are shown below. These are called **regular tessellations**.



As we saw in Problem 9.46, we have many more possibilities if we allow ourselves to use more than one type of polygon. Shown below are two **semiregular tessellations**, in which the same set of regular polygons surround each vertex in the tessellation. There are eight such tessellations (not including the regular tessellations); see if you can find the other six.

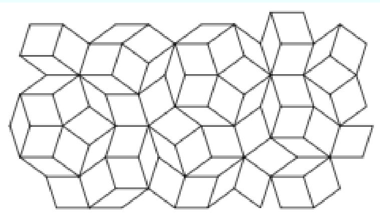


If we no longer restrict ourselves to having the same polygons around each point, we have **demiregular tessellations**, one of which is shown below.



3.4.6.4 / 4.6.12

Of course, we need not only use regular polygons in building tilings. Moreover, tilings need not be periodic (meaning having the same pattern over and over). **Penrose tilings** consist of two quadrilaterals that are used as tiles to make fascinating non-periodic tilings. An example is shown below.



We don't have to restrict ourselves to simple geometric figures to tile a whole plane. The Dutch artist M. C. Escher produced thousands of tessellations using various animal shapes and other whimsical designs. Rather than being satisfied with just a tessellation, Escher often wove his tessellations into very engaging pieces of art. You'll find links to Escher's art at `http://www.mcescher.com/`.

Tessellations have even made their way into puzzles and games, as the two pictures of the tessellation game *Bats and Lizards* below show. Perhaps you can see how these patterns were created from the regular tessellations on the previous page!